Ch10 Proof by Induction

- What it is
- Why it works
- How to do it.
- Augment boiler plate language from text.

Proof Cartoon

Prop: $\forall n \in \mathbb{N}$, $P(n)$.

Pf: (by induction on n)

(1) base step — Prove $P(1)$ is true.

(2) inductive step

inductive hypothesis

Prove that if $\boxed{P(k)}$ is true, then $P(k+1)$ is true for all $k \geq 1$.

[short-hand $P(k) \Rightarrow P(k+1)$]

Ex.

Prop $\forall n \in \mathbb{N}$

$$\boxed{1+2+\ldots+n = \frac{n(n+1)}{2}}.$$

$P(n)$

$P(1)$ is the statement

$$1 = \frac{1(1+1)}{2}.$$

Clearly $1 = \frac{1 \cdot 2}{2}$ ✓

If $1+2+\ldots+k = \frac{k(k+1)}{2}$, then

$$1+2+\ldots+(k+1) = \frac{(k+1)(k+2)}{2}.$$

Don't write this down. Prop: $\forall n \in \mathbb{N}$, $\overbrace{1+2+\cdots+n = \frac{n(n+1)}{2}}^{P(n)}$.

"Proof"

$\boxed{\begin{array}{l} P(1) \text{ is } 1 = \frac{1(2)}{2} \\ P(2) \text{ is } 1+2 = \frac{2(3)}{2} \end{array}}$ true,

check
$1+2=3 = \frac{3\cdot 2}{2} = \frac{2\cdot 3}{2}$ ✓

$P(3)$ is $1+2+3 = \frac{3(4)}{2}$.

check: $1+2+3=6 = \frac{12}{2} = \frac{3\cdot 4}{2}$ ✓

$P(4)$ is $1+2+3+4 = \frac{4(5)}{2}$.

check
$1+2+3+4 = 10 = \frac{2\cdot 2\cdot 5}{2} = \frac{4(5)}{2}$ ✓

$\vdots$

Alt. "proof"

$P(1)$ is true by checking num

$P(2)$ is true " "

$P(3)$

$1+2+3 = (1+2)+3 = \frac{2(3)}{2} + 3\cdot\frac{2}{2}$

$= \frac{3(2+2)}{2} = \frac{3(4)}{2}$ ✓

$P(4)$

$1+2+3+4 = (1+2+3)+4 = \frac{3\cdot 4}{2} + 4\cdot\frac{2}{2}$

$= \frac{4(3+2)}{2} = \frac{4(5)}{2}$

$P(5)$

$(1+2+3+4)+5 = \frac{4\cdot 5}{2} + 5\cdot\frac{2}{2} = \frac{5(6)}{2}$ ✓

Logical Structure

$P(1)$
$P(2)$
$P(3)$
$\vdots$] !!

$\therefore \forall n, P(n)$

Logical Structure

$P(1)$
$P(1) \Rightarrow P(2)$
$P(2) \Rightarrow P(3)$
$P(3) \Rightarrow P(4)$
$\vdots$] !!

$\therefore \forall n, P(n)$

$P(1)$
$\boxed{\forall k,\ P(k) \Rightarrow P(k+1)}$

$\therefore \forall n, P(n)$

Using the same argument in every case.

Prop: For all $n \in \mathbb{N}$, $\boxed{1+2+\dots+n = \frac{n(n+1)}{2}}$

Pf: (By induction on n)

(1) Let $n=1$. We need to show $1 = \frac{1(1+1)}{2}$.
But $1 = 1 \cdot \frac{2}{2} = \frac{1(1+1)}{2}$.

Inductive hypothesis

(2) We suppose $\boxed{1+2+\dots+k = \frac{k(k+1)}{2}}$ for some $k \in \mathbb{N}$. We must show
$1+2+\dots+(k+1) = \frac{(k+1)(k+2)}{2}$.

Observe

$$
\begin{aligned}
1+2+\dots+k+(k+1) &= (1+2+\dots+k)+(k+1) && \text{associativity} \\
&= \frac{k(k+1)}{2} + (k+1) && \boxed{\text{by the inductive hypothesis}} \\
&= \frac{k(k+1)+2(k+1)}{2} && \text{common denominator} \\
&= \frac{(k+1)(k+2)}{2} && \text{factor out } k+1;
\end{aligned}
$$

which is what we needed to show.

Since $\boxed{P(k)}$ implies $\boxed{P(k+1)}$ for all $k \geq 1$, we know the proposition holds for all $n \in \mathbb{N}$.

Prop: $\forall n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i(i!) = (n+1)! - 1$

Pf: (by induction on n)

(1) We need to check that $\displaystyle\sum_{i=1}^{1} i(i!) = (1+1)! - 1$.

Observe $\displaystyle\sum_{i=1}^{1} i(i!) = 1 \cdot (1!) = 1$ and $(1+1)! - 1 = 2! - 1 = 1$.

Thus, the base step holds.

(2) We suppose $\displaystyle\sum_{i=1}^{k} i(i!) = (k+1)! - 1$. We need to show that $\displaystyle\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1$.

Observe

$$\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)((k+1)!)$$

assoc

$$= (k+1)! - 1 + (k+1)((k+1)!)$$

by the inductive hypothesis

$$= (k+1)!(1 + k+1) - 1$$

factor out $(k+1)!$

$$= (k+2)! - 1$$