1. Two sets A and B have the same **cardinality** if $\exists\, f: A \to B$ bijection.

   Write $|A| = |B|$.

   Last time found $|\mathbb{N}| = |\mathbb{Z}|$, $|\mathbb{R}| = |(0,\infty)|$ $f(x) = e^x$

2. Theorem: $|\mathbb{N}| \neq |\mathbb{R}|$.

Proof. (by contradiction) Suppose $\exists$ bijection $f: \mathbb{N} \to \mathbb{R}$.

[Contradict the surjective property] Start listing the elements of $\mathbb{R}$ as images of f. (See table below)

Claim: We can find $x \in \mathbb{R}$ that is NOT in this table.

(i.e. $\nexists\, n \in \mathbb{N}$ s.t. $f(n) = x$. And thus f is NOT surjective.)

n	$f(n)$
1	0.4000000000000000...
2	8.5006070866690 0...
3	7.5050094004410 1...
4	5.5070400804805 0...
5	6.9002600000050 6...
6	6.8280958205002 0...
7	6.5050555065580 8...
8	8.7208064000044 8...
9	0.5500008888007 7...
10	0.5002072207805 1...
11	2.9000088000090 0...
12	6.5028000800967 1...
13	8.8900802400805 0...
14	8.5000874208022 6...
⋮	⋮

POC: Define x by defining each digit in its decimal representation.

Postion	Value	Ex
left of "."	0	0
10^{-1}	$\neq f(1)$'s 10^{-1} digit	$\neq 4$
10^{-2}	$\neq f(2)$'s 10^{-2} digit	$\neq 0$
10^{-3}	$\neq f(3)$'s 10^{-3} digit	$\neq 5$
		$\neq 0$
		$\neq 6$

we can do this using 0's + 1's only

$x = 0.01010\ldots$

Clearly $f(1) \neq x$ b/c they differ in 10^{-1}th position

$f(2) \neq x$ b/c they differ in 10^{-2} th position.

⋮

$f(n) \neq x$ b/c they differ in the 10^{-n} th position. □

So f is NOT onto. $\Rightarrow\!\!\!\!\not\;\;$ f is a bijection.

3. The set A is called **countably infinite** if $|A| = |\mathbb{N}|$.

   Countable means finite or countably infinite.

   $\mathbb{R}$ is uncountable.

4. Examples: $\mathbb{N}, \mathbb{Z}$ countable, $A = \{0,1,2\}$ countable

   $(0,\infty)$ uncountable.

5. Theorem 14.3: A set A is countably infinite if and only if its elements can be listed as an infinite sequence:

$$a_1, a_2, a_3, \ldots$$

$f: \mathbb{N} \to A$ defined as $f(n) = a_n$.

6. Theorem 14.4: $|\mathbb{Q}|$ is countably infinite.

Proof. Find a way to list elements of $\mathbb{Q}$.

	1	2	3	4	5	6	7	8	9	10		
	0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$
1	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	$\cdots$
2		$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	$\cdots$
3		$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	$\cdots$
4		$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	$\cdots$
5		$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	$\cdots$
6		$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	$\cdots$
7		$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Things to check.

1. Is every element of $\mathbb{Q}$ in this table?

$\boxed{\text{Yes}}$ If $q \in \mathbb{Q}$, then $q = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, and $\gcd(a,b) = 1$, and $b \in \mathbb{Z}^+$.

So q appears in col. w/ a as the column label and row at most b.

2. Will every element of $\mathbb{Q}$ eventually be on this list?

$\boxed{\text{Yes}}$ For $q = \frac{a}{b}$ (as above), q is in column label a which is at most column number $2|a|+1$. It is in row b. Let $k = \max\{2|a|+1, b+1\}$

Then q appears in the first $1 + k^2$ terms. □