

1. Two sets A and B have the same **cardinality** if

2. Theorem: $|\mathbb{N}| \neq |\mathbb{R}|$.

Proof.

| n | $f(n)$ |
|-----|-----------------------|
| 1 | 0.4000000000000000... |
| 2 | 8.5006070866690 0... |
| 3 | 7.5050094004410 1... |
| 4 | 5.5070400804805 0... |
| 5 | 6.9002600000050 6... |
| 6 | 6.8280958205002 0... |
| 7 | 6.5050555065580 8... |
| 8 | 8.7208064000044 8... |
| 9 | 0.5500008888007 7... |
| 10 | 0.5002072207805 1... |
| 11 | 2.9000088000090 0... |
| 12 | 6.5028000800967 1... |
| 13 | 8.8900802400805 0... |
| 14 | 8.5000874208022 6... |
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3. The set A is called **countably infinite** if

4. Examples:

5. Theorem 14.3: A set A is countably infinite if and only if

6. Theorem 14.4: $|\mathbb{Q}|$ is

Proof.

| | | | | | | | | | | | |
|---------------|---------------|----------------|----------------|-----------------|----------------|-----------------|----------------|-----------------|---------------|----------------|----------|
| 0 | 1 | -1 | 2 | -2 | 3 | -3 | 4 | -4 | 5 | -5 | ... |
| $\frac{0}{1}$ | $\frac{1}{1}$ | $\frac{-1}{1}$ | $\frac{2}{1}$ | $\frac{-2}{1}$ | $\frac{3}{1}$ | $\frac{-3}{1}$ | $\frac{4}{1}$ | $\frac{-4}{1}$ | $\frac{5}{1}$ | $\frac{-5}{1}$ | ... |
| | $\frac{1}{2}$ | $\frac{-1}{2}$ | $\frac{2}{3}$ | $\frac{-2}{3}$ | $\frac{3}{2}$ | $\frac{-3}{2}$ | $\frac{4}{3}$ | $\frac{-4}{3}$ | $\frac{5}{2}$ | $\frac{-5}{2}$ | ... |
| | $\frac{1}{3}$ | $\frac{-1}{3}$ | $\frac{2}{5}$ | $\frac{-2}{5}$ | $\frac{3}{4}$ | $\frac{-3}{4}$ | $\frac{4}{5}$ | $\frac{-4}{5}$ | $\frac{5}{3}$ | $\frac{-5}{3}$ | ... |
| | $\frac{1}{4}$ | $\frac{-1}{4}$ | $\frac{2}{7}$ | $\frac{-2}{7}$ | $\frac{3}{5}$ | $\frac{-3}{5}$ | $\frac{4}{7}$ | $\frac{-4}{7}$ | $\frac{5}{4}$ | $\frac{-5}{4}$ | ... |
| | $\frac{1}{5}$ | $\frac{-1}{5}$ | $\frac{2}{9}$ | $\frac{-2}{9}$ | $\frac{3}{7}$ | $\frac{-3}{7}$ | $\frac{4}{9}$ | $\frac{-4}{9}$ | $\frac{5}{6}$ | $\frac{-5}{6}$ | ... |
| | $\frac{1}{6}$ | $\frac{-1}{6}$ | $\frac{2}{11}$ | $\frac{-2}{11}$ | $\frac{3}{8}$ | $\frac{-3}{8}$ | $\frac{4}{11}$ | $\frac{-4}{11}$ | $\frac{5}{7}$ | $\frac{-5}{7}$ | ... |
| | $\frac{1}{7}$ | $\frac{-1}{7}$ | $\frac{2}{13}$ | $\frac{-2}{13}$ | $\frac{3}{10}$ | $\frac{-3}{10}$ | $\frac{4}{13}$ | $\frac{-4}{13}$ | $\frac{5}{8}$ | $\frac{-5}{8}$ | ... |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \ddots |

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