My solutions

1. Proof by Contrapositive

Proposition: If P, then Q. **Proof:** (by contrapositive) Suppose $\sim Q$. $\vdots$ Thus, $\sim P$. $\square$

2. Prove that for every integer n, if $n^2 + 3n$ is odd, then n is odd.

   Proof by Contrapositive. We prove the contrapositive: If n is even, then $n^2 + 3n$ is even.

   Assume n is even. Then $n = 2k$ for some integer k.

   We have:

   $$\begin{aligned} n^2 + 3n &= (2k)^2 + 3(2k) \\ &= 4k^2 + 6k \\ &= 2(2k^2 + 3k) \end{aligned}$$

   Since $2k^2 + 3k$ is an integer, we see that $n^2 + 3n = 2(2k^2 + 3k)$ is even.

   Therefore, by contrapositive, if $n^2 + 3n$ is odd, then n is odd. $\square$

3. Prove that for every pair of real numbers x and y, if $x + y$ is irrational, then either x is irrational or y is irrational.

   Proof by Contrapositive. We prove the contrapositive: If both x and y are rational, then $x + y$ is rational.

   Assume x and y are both rational. Then $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$.

   We have:

   $$\begin{aligned} x + y &= \frac{a}{b} + \frac{c}{d} \\ &= \frac{ad + bc}{bd} \end{aligned}$$

   Since a, b, c, d are integers, we know that $ad + bc$ and bd are integers, and $bd \neq 0$ (since $b \neq 0$ and $d \neq 0$).

   Therefore $x + y = \frac{ad+bc}{bd}$ is rational.

   Hence, by contrapositive, if $x + y$ is irrational, then either x is irrational or y is irrational. $\square$

4. Use proof by contrapositive to prove each statement below.

(a) If the product of two integers ab is even, then a is even or b is even.

Proof by Contrapositive. We prove the contrapositive: If a is odd and b is odd, then ab is odd. Assume a and b are both odd. Then $a = 2m+1$ and $b = 2n+1$ for some integers m and n. We have:

$$\begin{aligned} ab &= (2m+1)(2n+1) \\ &= 4mn+2m+2n+1 \\ &= 2(2mn+m+n)+1 \end{aligned}$$

Since $2mn+m+n$ is an integer, we see that $ab = 2(2mn+m+n)+1$ is odd.
Therefore, by contrapositive, if ab is even, then a is even or b is even. $\square$

(b) If n^2 is a multiple of 3, then n is a multiple of 3.

Proof by Contrapositive. We prove the contrapositive: If n is not a multiple of 3, then n^2 is not a multiple of 3.
Assume n is not a multiple of 3. By the division algorithm, we can write $n = 3q+r$ where q is an integer and $r \in \{0,1,2\}$. Since n is not a multiple of 3, we have $r \neq 0$, so $r \in \{1,2\}$.
Case 1: $r = 1$, so $n = 3q+1$.
Then:

$$\begin{aligned} n^2 &= (3q+1)^2 \\ &= 9q^2+6q+1 \\ &= 3(3q^2+2q)+1 \end{aligned}$$

Since $3q^2+2q$ is an integer, $n^2 = 3(3q^2+2q)+1$ leaves a remainder of 1 when divided by 3, so n^2 is not a multiple of 3.
Case 2: $r = 2$, so $n = 3q+2$.
Then:

$$\begin{aligned} n^2 &= (3q+2)^2 \\ &= 9q^2+12q+4 \\ &= 9q^2+12q+3+1 \\ &= 3(3q^2+4q+1)+1 \end{aligned}$$

Since $3q^2+4q+1$ is an integer, $n^2 = 3(3q^2+4q+1)+1$ leaves a remainder of 1 when divided by 3, so n^2 is not a multiple of 3.
In both cases, n^2 is not a multiple of 3.
Therefore, by contrapositive, if n^2 is a multiple of 3, then n is a multiple of 3. $\square$

(c) Suppose $x \in \mathbb{R}$. If $x^7 - 3x^4 + 10x^3 - x^2 - \pi \geq 0$, then $x \geq 0$.

Proof by Contrapositive. We prove the contrapositive: If $x < 0$, then $x^7 - 3x^4 + 10x^3 - x^2 - \pi < 0$.

Assume $x < 0$. We analyze the sign of each term in the expression $x^7 - 3x^4 + 10x^3 - x^2 - \pi$:

- $x^7 < 0$ (negative number to an odd power is negative)
- $-3x^4 < 0$ (since $x^4 > 0$ for $x \neq 0$, so $-3x^4 < 0$)
- $10x^3 < 0$ (negative number to an odd power is negative, and $10 \cdot (\text{negative}) < 0$)
- $-x^2 < 0$ (since $x^2 > 0$ for $x \neq 0$, so $-x^2 < 0$)
- $-\pi < 0$ (since $\pi > 0$)

Therefore:

$$x^7 - 3x^4 + 10x^3 - x^2 - \pi = (\text{negative}) + (\text{negative}) + (\text{negative}) + (\text{negative}) + (\text{negative}) < 0$$

Hence, by contrapositive, if $x^7 - 3x^4 + 10x^3 - x^2 - \pi \geq 0$, then $x \geq 0$. $\square$