1. Tautologies and Contradictions

   Always true: $P \vee \sim P$, $P \Rightarrow P$

   Always false: $P \wedge \sim P$, $(P \Leftrightarrow Q) \wedge (P \wedge \sim Q)$

2. Proof by Contradiction

Proposition: P is true. **Proof:** (by contradiction) Suppose $\sim P$. $\vdots$ Thus, $C \wedge \sim C$. $\square$	**Proposition:** If P, then Q. **Proof:** (by contradiction) Suppose P and $\sim Q$. $\vdots$ Thus, $C \wedge \sim C$. $\square$

3. Is this a valid argument?
   When $P =$ true, $\sim P \Rightarrow (C \wedge \sim C)$ is true. When $P =$ false, $\sim P \Rightarrow (C \wedge \sim C)$ is false. So, yes, it is a valid argument.

4. Prove that $\sqrt{2}$ is irrational.

   Proof. (by contradiction) Suppose $\sqrt{2}$ is rational. By the definition of rational, this implies that there exist integers a and b such that $\sqrt{2} = \frac{a}{b}$. We choose a representation, $\frac{a}{b}$, that is in *lowest terms.*

   By squaring both sides of $\sqrt{2} = \frac{a}{b}$, we obtain $2 = \frac{a^2}{b^2}$, or, equivalently $2b^2 = a^2$.

   The last equality implies that a^2 is even. Since the square of odd numbers is odd, it follows that a is even. Thus, there is an integer k such that $a = 2k$. Returning to the expression $2b^2 = a^2$ and replacing a with $2k$, we obtain $2b^2 = 4k^2$. Dividing the previous equality by 2 gives the equation $b^2 = 2k^2$ which implies that b^2 is even. Thus, we conclude that b is even. Thus, $\frac{a}{b}$ is a ratio of even numbers and not in lowest terms.

   Now we have the contradiction that the expression $\frac{a}{b}$ is in lowest terms and not in lowest terms. So $\sqrt{2}$ is not rational. Thus, we conclude $\sqrt{2}$ is irrational. $\square$

5. Use proof by contradiction.

   (a) Prove if $a, b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.

   Proof. (by contradiction) Suppose $a, b \in \mathbb{Z}$ and $a^2 - 4b - 3 = 0$. Thus, $a^2 = 4b + 3$. The last equality implies that a^2 is odd and therefore a is odd. Thus, there exists an integer k such that $a = 2k + 1$.

   Now, substituting $a = 2k + 1$ into $a^2 = 4b + 3$, we obtain $4k^2 + 4k + 1 = 4b + 3$. Let $n = 4k^2 + 4k + 1 = 4b + 3$. Observe that $4k^2 + 4k + 1 \equiv 1 \pmod{4}$ but $4b + 3 \equiv 3 \pmod{4}$. Thus, we have the contradiction that, when divided by 4, the integer n has a remainder of 1 and a remainder of 3.

   Thus, it is not possible for two integers to satisfy the expression $a^2 - 4b - 3 = 0$. Thus, if $a, b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$. $\square$

   (b) Prove that for every $x \in [\frac{\pi}{2}, \pi]$, $\sin(x) - \cos(x) \geq 1$.

   Proof. (by contradiction) Suppose that there exists an $x \in [\frac{\pi}{2}, \pi]$ such that $\sin(x) - \cos(x) < 1$. In the interval $[\frac{\pi}{2}, \pi]$, we know $\sin(x) \geq 0$ and $\cos(x) \leq 0$. Thus, we know that $\sin(x) - \cos(x) \geq 0$. Since $0 \leq \sin(x) - \cos(x) < 1$, we also know that $0 \leq (\sin(x) - \cos(x))^2 < 1$.

   On the other hand, we know that $\sin^2(x) + \cos^2(x) = 1$ from trigonometry and $-2\sin(x)\cos(x) \geq 0$ because $\sin(x) \geq 0$ and $\cos(x) \leq 0$. Now, we observe that

   $$(\sin(x) - \cos(x))^2 = \sin^2(x) + \cos^2(x) - 2\sin(x)\cos(x) \geq 1.$$

   This lead to the contradiction that $(\sin(x) - \cos(x))^2$ is both strictly less than 1 and greater than or equal to 1. Thus, it is not possible for there to be an x-value in $[\frac{\pi}{2}, \pi]$ such that $\sin(x) - \cos(x) < 1$. Thus, for every $x \in [\frac{\pi}{2}, \pi]$, $\sin(x) - \cos(x) \geq 1$.

   $\square$