Prove the following statements. Use any method you like, but follow directions.

1. Given an integer $a$, then $a^2+4a+7$ is odd if and only if $a$ is even.

   *Proof.* ($\Rightarrow$:) (by contrapositive) Suppose $a$ is odd. Then by the definition of odd, we know there exist an integer $k$ such that $a = 2k+1$. Substituting into $a^2+4a+7$, we obtain

   $$a^2+4a+7 = (2k+1)^2+4(2k+1)+7 = 4k^2+4k+1+8k+4+7 = 4k^2+12k+12 = 2(2k^2+6k+6),$$

   where $2k^2+6k+6$ is an integer. Thus, $a^2+4a+7$ is even.

   We have shown that if $a$ is odd, then $a^2+4a+7$ is even which this is equivalent to the statement that if $a^2+4a+7$ is odd, then $a$ is even.

   ($\Leftarrow$:) (direct) Suppose $a$ is even. Then $a = 2k$, for some integer $k$. Substituting into $a^2+4a+7$, we obtain

   $$(2k)^2+4(2k)+7 = 4k^2+8k+2\cdot 3+1 = 2(2k^2+4k+3)+1,$$

   where $2k^2+4k+3$ is an integer. Thus, $a^2+4a+7$ is odd. $\square$

2. There exists a set $X$ such that $\mathbb{N} \in X$ and $N \subseteq X$.

   *Proof.* Let $X = \mathbb{N} \cup \{\mathbb{N}\} = \big\{\{\mathbb{N}\}, 1, 2, 3, \ldots\big\}$.
   We can see that $\mathbb{N} \in X$ since in the "list" form of $X$ on the right, $\mathbb{N}$ is the first element in the list. We can see that $\mathbb{N} \subseteq X$ because we see that for every $n \in \mathbb{N}$, $n \in X$.

   $\square$

3. Suppose $x, y \in \mathbb{R}$. Then $(x+y)^2 = x^2+y^2$ if and only if $x = 0$ or $y = 0$.

   *Proof.* Let $x, y \in \mathbb{R}$.
   ($\Rightarrow$:) (direct) Suppose $(x+y)^2 = x^2+y^2$. By expanding and rearranging $(x+y)^2 = x^2+y^2$ we obtain $2xy = 0$. Now, we apply a property of real numbers that if a product is zero, at least one of its terms is zero. Thus, 2, $x$ or $y$ is zero. But 2 is not zero. So either $x = 0$ or $y = 0$.

   ($\Leftarrow$:) (direct) Suppose $x = 0$ or $y = 0$.
   If $x = 0$, then by substitution the expression $(x+y)^2 = x^2+y^2$ becomes $y^2 = y^2$, which is always true. The argument of $y = 0$ is the same. Thus, if $x = 0$ or $y = 0$, then $(x+y)^2 = x^2+y^2$. $\square$

4. Suppose $a, b, c \in \mathbb{N}$. Use the proposition we proved in class to show that if $a \big| bc$ and $\gcd(a,b) = 1$, then $a \big| c$.

*Proof.* Suppose $a,b,c \in \mathbb{N}$, $a \big| bc$ and $\gcd(a,b) = 1$.
Since $a \big| bc$, there exists an integer $k$ such that $ak = bc$. Since $\gcd(a,b) = 1$, by Prop. 7.1, there exist integers $m$ and $n$ such that $an + bm = 1$.

Multiply the previous equation by $c$ to obtain $anc + bmc = c$. Using the fact that $ak = bc$, we can plug in for $bc$ into $amc + bmc = c$ to obtain $c = anc + akm = a(nc + km)$.

Now, $c = a(nc + km)$ where $nc + km$ is an integer. Thus, $a \big| c$ by definition, which is what we wanted to prove.

$\square$