MATH 265: Introduction to Mathematical Proofs **Worksheet 15: Ch 8**

Provide a **formal, complete** proof for each statement below.

1. $\{12a+25b : a,b \in \mathbb{Z}\} = \mathbb{Z}$.

   Proof. We will proceed by mutual containment.

   ($\subseteq$:) Let $x \in \{12a+25b : a,b \in \mathbb{Z}\}$. Thus, there exist integers a and b such that $x = 12a+25b$. Thus, x is a product and sum of integers, x is an integer. So, $x \in \mathbb{Z}$.

   ($\supseteq$:) Let $x \in \mathbb{Z}$.
   Since $gcd(12,25) = 1$, Proposition 7.1 implies that there exist integers k and ℓ such that

   $$1 = 12k+25\ell.$$

   Now, multiply the previous equality by x to get

   $$x = 12(kx)+24(\ell x).$$

   Since k, ℓ, and x are all integers, kx and ℓx are integers. Thus, we have written an arbitrary integer x in the form $12a+25b$ where $a = kx$ and $b = \ell x$, both integers. Thus, we have shown that $x \in \{12a+25b : a,b \in \mathbb{Z}\}$. □

2. If A and B are sets in a universal set U, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

   Proof. Suppose A and B are sets in a universal set U. We will proceed by mutual containment.

   ($\subseteq$:) Let $x \in \overline{A \cup B}$. Then x is not in $A \cup B$. Thus, x cannot be in A and x also cannot be in B. Since x is not in A, $x \in \overline{A}$. Since x is not in B, $x \in \overline{B}$. Thus, we have shown that $x \notin A$ and $x \notin B$. Thus, $x \in \overline{A} \cap \overline{B}$.

   ($\supseteq$:) Let $x \in \overline{A} \cap \overline{B}$. Thus, $x \in \overline{A}$ and $x \in \overline{B}$. This last statement is equivalent to$x \notin A$ and $x \notin B$. Since x is in neither A not B, x cannot be in $A \cup B$. Thus, $x \in \overline{A \cup B}$.

   □

3. For every pair of sets A and B, $A \subseteq B$ if and only if $A - B = \emptyset$.

   Proof. Let A and B be sets. We will prove the equivalent statement that $A \not\subseteq B$ if and only if $A - B \neq \emptyset$. (This is effectively a "double" contrapositive.)

   ($\Rightarrow$:) Suppose $A \not\subseteq B$. Then there exists $a \in A$ such that $a \notin B$. Thus, $a \in A - B$. Thus, $A - B\ \not\emptyset$.

   ($\Leftarrow$:) Suppose $A - B \neq \emptyset$. Thus, there exists some $x \in A - B$. Since $x \in A - B$, it follows that $x \in A$ and $x \notin B$. Thus, it is not the case that every element of x is an element of B. Thus, $A \not\subseteq B$.

   □