Solutions

1. Disproving the statement P means proving $\sim P$.

2. Disprove each false statement below by **first** stating what you need to show and then providing a complete disproof.

   (a) If $n \in \mathbb{Z}$ and $n^5 - 3n$ is even, then n is even.
   Answer: This is false. We must find an integer n such that $n^5 - 3n$ is even, but n is odd.

   Pick $n = 1$. We that n is integer and $n^5 - 3n = -2$ which is even. But n is odd.

   (b) There exists a real number such that $x^2 < x < |x|$.
   Answer: This is false. We must show that for every integer x either $x \leq x^2$ or $x \geq |x|$.
   We will proceed by cases.
   Case 1: Suppose $x \geq 0$. Then $x \geq |x|$, since $x = |x|$ for positive numbers.
   Case 1: Suppose $x < 0$. Since $x^2 \geq 0$ for all real numbers, it follows that $x < 0 \leq x^2$. Thus, $x \leq x^2$.

   (c) For all sets A and B, if $A - B = \emptyset$, then $B \neq \emptyset$.
   Answer: This is false. We must find a pair of sets A and B such that $A - B = \emptyset$ and $B = \emptyset$.
   Pick $A = B = \emptyset$. We see that A and B are sets and $A - B = \emptyset$ and $B = \emptyset$.

3. Prove or disprove the statements below. Your disproof must be preceded by a description of what you must show in order to disprove the statement.

   (a) Every odd integer is the sum of three odd integers.
   Answer: This is true.

   Proof. Let n be an odd integer. Since n is odd, so is the integer $-n$. Now, $n = n + n + (-n)$ a sum of three odd integers. □

   (b) If A and B are sets, then $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A - B)$.
   Answer: This is false. We must find two sets A and B such that $\mathscr{P}(A) - \mathscr{P}(B) \not\subseteq \mathscr{P}(A - B)$. Alternatively, we need to find A and B such that there exists some $X \in \mathscr{P}(A) - \mathscr{P}(B)$ such that $X \notin \mathscr{P}(A - B)$.
   Pick $A = \{1, 2\}$ and $B = \{2\}$. Thus, $A - B = \{1\}$. Choose, $X = A$. We know that $A \in \mathscr{P}(A)$ for all sets A. Thus, $\{1, 2\} \in \mathscr{P}(A)$. Since $1 \notin B$, then $A \notin \mathscr{P}(B)$. So, $A \in \mathscr{P}(A) - \mathscr{P}(B)$. Since $2 \notin A - B$, $A \notin \mathscr{P}(A - B)$.

   (c) For all $a, b, c \in \mathbb{N}$, if $a \mid bc$, then $a \mid b$ or $a \mid c$.
   Answer: This is false. We need to find integers a, b, and c such that $a \mid bc$ and $a \nmid b$ and $a \nmid c$.
   Pick $a = 6$, $b = 2$ and $c = 3$. Then $a \mid bc$ since $6 \mid 6$. But $a \nmid b$ and $a \nmid c$ since $6 \nmid 2$ and $6 \nmid 3$.