1. Prove the statement below using induction.

   The expression n^3+5n+6 is divisible by 3 for every $n \in \mathbb{N}$.

   Proof. We proceed by induction on n.

   (1) **Base case ($n=1$):** In this case, $n^3+5n+6=12$ which is divisible by 3. Hence, the statement holds for $n=1$.

   (2) **Inductive step:** Assume that for some $k \geq 1$, k^3+5k+6 is divisible by 3. Thus, $k^3+5k+6=3m$, for some integer m.

   We must show that $(k+1)^3+5(k+1)+6$ is divisible by 3.

   Observe

$$\begin{aligned}(k+1)^3+5(k+1)+6 &= (k^3+3k^2+3k+1)+(5k+5)+6 && \text{by expansion}\\ &= (k^3+5k+6)+(3k^2+3k+6) && \text{collecting terms}\\ &= 3m+3(k^2+k+2) && \text{inductive hypothesis, factoring}\\ &= 3(m+k^2+k+2).\end{aligned}$$

   Since $(k+1)^3+5(k+1)+6=3(m+k^2+k+2)$ where $m+k^2+k+2 \in \mathbb{Z}$, it follows that 3 divides $(k+1)^3+5(k+1)+6$. □

2. Can you think of any **other** ways one might prove the statement above?
   Try by cases: $n=3\ell$, $n=3\ell+1$ and $n=3\ell+2$.

3. Proof by **Strong** Induction

   Propostion: For every $n \in N$, $P(n)$.

   proof: (by induction on n)
   (1) Base Step: Prove as many instances $P(1), P(2), \ldots$ in order to make your induction step work.
   (2) Inductive Step: Suppose that for some $k \in \mathbb{N}$, all statements $P(1), P(2), \cdots, P(k)$ are true. Show $P(k+1)$ is true.

4. Proposition: Let a_n be the sequence such that $a_0=1$, $a_1=4$ and $a_n=3a_{n-1}-2a_{n-2}$. Then, for every $n \in \mathbb{N} \cup \{0\}$, $a_n = 3 \cdot 2^n - 2$.

   First, make sure you understand the statement and crunch a few numbers.

   Proof. We proceed by induction on n.

   (1) **Base cases ($n=0$ and $n=1$):** We know $a_0=1$ and $a_1=4$ by definition. We check that the closed formula gives the same values:

$$a_0 = 3 \cdot 2^0 - 2 = 1 \text{ and } a_1 = 3 \cdot 2^1 - 2 = 4.$$

Thus, the proposition holds for $n = 0$ and $n = 1$.

(2) **Inductive step:** Assume that for some $k \geq 1$, the term a_i is equal to $3 \cdot 2^i - 2$ for all $0 \leq i \leq k$. We must show that the term a_{k+1} is equal to $3 \cdot 2^{k+1} - 2$.

Observe

$$\begin{aligned}
a_{k+1} &= 3a_k - 2a_{k-1} && \text{definition of } a_{k+1} \\
&= 3(3 \cdot 2^k - 2) - 2(3 \cdot 2^{k-1} - 2) && \text{inductive hypothesis for } k \text{ and } k-1 \\
&= 9 \cdot 2^k - 6 - 2 \cdot 3 \cdot 2^{k-1} + 4 && \text{expansion} \\
&= 9 \cdot 2^k - 3 \cdot 2^k - 6 && \text{combine terms} \\
&= 6 \cdot 2^k - 6 && \text{combine terms} \\
&= 3 \cdot 2^{k+1}k - 6 && ,
\end{aligned}$$

$whichiswhatweneededtoshow.$

☐

5. Proposition: Every integer $n > 1$ has a prime factorization.

Proof. We proceed by induction on n.

(1) **Base cases ($n = 2$):** The integer 2 is a prime and this is its own prime factorization.

(2) **Inductive step:** Assume that for some $k \geq 2$, the integer i has a prime factorization for all $2 \leq i \leq k$. We must show that the integer $k+1$ has a prime factorization. We will proceed by cases.

Case 1: Suppose the integer $k+1$ is prime.
Then $k+1$ is its own prime factorization.

Case 2: Suppose the integer $k+1$ is not prime.
Then $k+1$ is composite and therefore can be written as a product of two integers, say a and b such that $2 \leq a < k+1$ and $2 \leq b < k+1$. Since a and b are both integers strictly less than $k+1$ and at least 2, the inductive hypothesis applies to each of them. Thus, both a and b have prime factorizations. Since $k+1 = ab$, it follows that the prime factorizations of a and b together form a prime factorization of $k+1$. ☐

(Bonus:) Any postage, n, can be made from 3-cent and 7-cent stamps provided $n \in \mathbb{N}$ and $n > 12$.