1. A **relation** R on A is a subset of $A \times A$.

   Notation: The words " a relates to b" or "the ordered pair (a,b) is in the relation can be written $(a,b) \in R$ or $a\,R\,b$. The same for **not** in R.

2. Examples

   (a) R is a relation on $\mathbb{Z}$ where $R = \{(0,0),(0,5),(-3,7),4,17)\}$.

   (b) R is a relation on $\mathbb{Z}$ where $(a,b) \in R$ if $a \leq b-2$

   (c) R is a relation on $\mathbb{R}$ where $(x,y) \in R$ if $x^2 = y$

   (d) R is a relation on $\mathbb{N} \times \mathbb{N}$ where $(a,b)\,R\,(c,d)$ if $ad = bc$.

3. Suppose R is a relation on the set A.

   (a) We say R is **symmetric** if $(a,b) \in R$ implies $(b,a) \in R$.

   (b) We say R is **reflexive** if for every $a \in A$, $(a,a) \in R$.

   (c) We say R is **transitive** if $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$.

4. Let R be the relation on $A = \{0,1,2,3,4\}$ defined as $a\,R\,b$ if $a \mid b$. List all the elements in R. What is the smallest positive number you could add to the set A that would increase the order R as much as possible?

   $R = \{(0,0),(1,0),(2,0),(3,0),(4,0),(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

   By inspection, it is reflexive and transitive. It is not symmetric since $(0,1) \in R$ but $(1,0) \notin R$.

   Add 12 since all elements of A will divide it.

5. Let R be a relation on $A = \mathbb{Z}$ defined by $a\,R\,b$ if $a \equiv b \pmod 3$. Find three elements of R that contain zero in the first coordinate, find three elements of R that contain 1 in the second coordinate, and find three ordered pairs in $\mathbb{Z} \times \mathbb{Z}$ that are not in R.

   Some things in R: $(0,3),(0,-3),(0,18),(4,1),(19,1),(-2,1)$

   Not in R: $(0,1),(4,2),(12,98)$

   Reflexive: Since $3 \mid 0$, for every $n \in Z$, $3 \mid (n-n)$. Thus, for every $n \in \mathbb{Z}$, $n \equiv n \pmod 3$. Thus, for every n, $(n,n) \in R$.

   Symmetric: Suppose $(a,b) \in R$. Then $a \equiv b \pmod 3$. So $3 \mid (a-b)$ or, equivalently there exists some integer k such that $ck = a-b$. Now, $c(-k) = b-a$, where $-k$ is an integer. So, $c \mid (b-a)$. So $(b,a) \in R$.

   Transitive: Suppose $(a,b) \in R$ and $(b,c) \in R$. Thus, there are integers k and ℓ such that $3k = a-b$ and $3\ell = b-c$. Adding the previous two equations gives us $3(k+\ell) = a-c$. Thus, $3 \mid (a-c)$ and $(a,c) \in R$.

6. Suppose R is a relation on $\mathscr{P}(A)$ where $A = \{0,1,2,3,4\}$ defined as $X\,R\,Y$ if $X \cap Y \neq \emptyset$.

   Some things in $R:$ $(\{0,1\},\{1,2\}),(A,A),(\{0,1\},\{1,2,3\}),(\{2,3,4\},\{1,2,3\}),(A,\{1,2,3\})$

   Not in R.: $(\emptyset,A),(\{0\},\{1\}),(\{1,2,3\},\{0,4\})$

   Reflexive: No. $(\emptyset,\emptyset) \notin R$

   Symmetric: Yes. Suppose $(X,Y) \in R$. Then, $X \cap Y \neq \emptyset$. Since $X \cap Y = Y \cap X$, it follows $Y \cap X \neq \emptyset$ and hence $(Y,X) \in R$.

   Transitive: No. Consider the elements $X = \{0,1\}, Y = \{1,2\}$ and $Z = \{2,3\}$. We see that $X \cap Y$ and $Y \cap Z$ are both nonempty but $X \cap Z = \emptyset$. So, $(X,Y),(Y,Z) \in R$ but $(X,Y) \notin R$.