1. Quick Review
   (a) R is a relation on A if $R \subseteq A \times A$
   (b) Suppose R is a relation on the set A.
      i. We say R is **symmetric** if $\forall a,b \in A \quad (a,b) \in R \Rightarrow (b,a) \in R$.
      ii. We say R is **reflexive** if $\forall a \in A, \ (a,a) \in R$
      iii. We say R is **transitive** if $\forall a,b,c \in A \quad (a,b),(b,c) \in R \Rightarrow (a,c) \in R$.
   (c) What properties do the following relations have?
      i. Let R be a relation on $\mathbb{N} \times \mathbb{N}$ such that $(a,b)R(c,d)$ if $ad = bc$. ($\frac{a}{b} = \frac{c}{d}$)
      symmetric
      reflexive
      transitive
      $[(1,1)] = \{ \ldots (1,1), (2,2), (3,3) \ldots \} = [(2,2)]$
      $[(1,2)] = \{ (1,2), (2,4), (3,6), (4,8) \ldots \}$
      $[(2,3)] = \{ (2,3), (4,6), (6,9), (8,12) \ldots \}$
      ii. Let R be a relation on $A = \mathbb{Z}$ defined by $a\,R\,b$ if $a \equiv b \pmod 3$.
      symmetric
      reflexive
      transitive
      $[3] = [0] = \{ a \in \mathbb{Z} : a \equiv 0 \bmod 3 \} = \{ \ldots -6, -3, 0, 3, 6, 9, \ldots \}$
      $[-2] = [1] = \{ \ldots -5, -2, 1, 4, 7, \ldots \}$
      $[8] = [2] = [\ldots -4, -1, 2, 5, 8, \ldots]$
      iii. Suppose R is a relation on $\mathscr{P}(A)$ where $A = \{0,1,2,3,4\}$ defined as $X\,R\,Y$ if $X \cap Y \neq \emptyset$.
      not symmetric $\rightsquigarrow (\emptyset, \emptyset) \notin R$
      reflexive
      not transitive
      $\{1,2\} R \{2,3\}$ and $\{2,3\} R \{3,4\}$
      but $\{1,2\} \not R \{3,4\}$

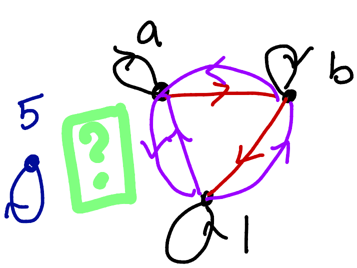
2. A relation R on set A is an **equivalence relation** if it's symmetric, reflexive, and transitive.
   - $[a]$ = equivalence class containing $a \in A$ $= \{x \in A : x \, R \, a\}$
   - A partition of a set S is a set of subsets of S that are ① nonempty ② whose union is all of S and ③ intersection of different subsets is empty.
   - The equivalence relation R partitions the set S into equivalence classes.

3. Build your own equivalence relation on $A = \{a, b, 1, 5\}$.

Now we have ALL possible ordered pairs w/ a, b, 1 ...!!

$R = \{ (a,a), (b,b), (1,1), (5,5), (a,b), (b,1), (b,a), (1,b), (a,1) \ (1,a) \}$

mandatory — just for kicks — symmetry forces this — transitivity



← our work is indep. of any details about A or R beyond the defn. of equivalence relation.

A is partitioned into $\{a,b,1\}$ $\{5\}$

4. For each relation below, quickly confirm the relation **is** an equivalence relation, then identify its equivalence classes using the square bracket notation **in two different ways** and by describing them as sets. Can you describe the partition of A that is produced?

   (a) Let R be a relation on $A = \mathbb{Z}$ defined by $a\,R\,b$ if $a \equiv b \pmod 6$.

   $[0] = [6] = \{\dots -6, 0, 6, 12, \dots\}$ $[3] = [9] = \{\dots 3, 9, 12, \dots\}$
   $[1] = [7] = \{\dots -5, 1, 7, 13, \dots\}$ $[4] = [10] = \{\dots -2, 4, 10, 16, \dots\}$
   $[2] = [8] = \{\dots, -4, 2, 8, 14, \dots\}$ $[5] = [11] = \{\dots, -1, 5, 11, 17, \dots\}$

   (b) Let R be a relation on the set of all polynomials with real coefficients defined by $p(x)Rq(x)$ if the degree of $p(x)$ equals the degree of $q(x)$.

   $[1] = [2] = \{p(x) = c : c \in \mathbb{R}\}$
   $[x] = [x+1] = \{ax+b : a, b \in \mathbb{R}, a \neq 0\}$
   $[x^2] = [10x^2] = \{ax^2+bx+c : a, b, c \in \mathbb{R}, a \neq 0\}$
   ⋮

   polynomial are partitioned according to the highest power of x.

   (c) Let R be the set of ~~continuous~~ differentiable functions $f : \mathbb{R} \to \mathbb{R}$ defined by by $p(x)Rq(x)$ if $p'(x) = q'(x)$.

   $[x] = [x+5] = \{x + c : c \in \mathbb{R}\}$ ← $\int 1\,dx$
   $[3x] = [3x+5] = \{3x + c : c \in \mathbb{R}\}$ ← $\int 3\,dx$
   $[\sin x] = [\sin(x) + \pi] = \{\sin(x) + c : c \in \mathbb{R}$ ← $\int \cos(x)\,dx$
   ⋮

   Differentiable fcns are partitioned into families of antiderivatives.

5. Suppose R and S are two equivalence relations on A. Answer the following questions **rigorously**.

   (a) Is the set $R \cup S$ an equivalence relation on A? No. Let $A = \{1,2,3\}$. Suppose $R = \{(1,1), (2,2), (3,3), (1,2)(2,1)\}$ and $S = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

   Then $R \cup S$ will contain $(1,2)$ and $(2,3)$ but not $(1,3)$.

   (b) Is the set $R \cap S$ an equivalence relation on A? Yes.

   reflexive: $\forall a \in A$, aRa and aSa b/c R and S are reflexive. Thus $(a,a) \in R \cap S$.

   Symmetric: Let $(a,b) \in R \cap S$. So $(a,b) \in R$ and $(a,b) \in S$. Since R and S are symmetric, $(b,a) \in R$ and $(b,a) \in S$. Thus, $(b,a) \in R \cap S$.

   reflexive: Let $(a,b), (b,c) \in R \cap S$. So $(a,b), (b,c) \in R$ and $(a,b), (b,c) \in S$. Since R and S are transitive, $(a,c) \in R$ and $(a,c) \in S$. So $(a,c) \in R \cap S$.