MATH 265: Introduction to Mathematical Proofs — Worksheet 5: §2.4-2.6

Jill's Solutions

1. Review:

   (a) Fill out the truth table for the biconditional statement.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

   (b) For **objects** A and B, how could you show that the statement below was **false**.

   $AB = 0$ if and only if $A = 0$ or $B = 0$.

   ans Find two objects A and B such that $AB = 0$ and $A \ \emptyset$ and $B \ \emptyset$. Thus, $P : AB = 0$ is true and $Q : A = 0 \vee B = 0$ is false.

   FYI: The biconditional statement is true for real numbers but false, in general, for matrices, modular arithmetic, and many other mathematical objects.

   (c) Fill out DeMorgan's Laws

   $\sim (P \vee Q)$ is equivalent to $\sim P \wedge \sim Q$

   $\sim (P \wedge Q)$ is equivalent to $\sim P \vee \sim Q$

2. Use a truth table to demonstrate that $\boxed{P \Rightarrow Q}$ is equivalent to $\boxed{\sim P \vee Q}$.

1	2	3	4	5
P	Q	$P \Rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

   We see that $\boxed{P \Rightarrow Q}$ is equivalent to $\boxed{\sim P \vee Q}$ because columns 3 and 5 are the same.

3. Use the equivalence above to rewrite the conditional statement in an equivalent form.

   If $f'(a) = 0$, then $f(a)$ is a maximum.

   $f'(a) \neq 0$ or $f(a)$ is a maximum.

4. Prove that $\boxed{\sim (P \Rightarrow Q)}$ is equivalent to $\boxed{P \wedge \sim Q}$ by constructing a string of logical equivalences that start with $\sim (P \Rightarrow Q)$ and end with $P \wedge \sim Q$. Each step must be justified by a specific, already established and referenced, logical equivalence.

   Proof:

$$\begin{aligned} \sim (P \Rightarrow Q) &= \sim (\sim P \vee Q) && \text{by previous problem (\# 2 above)} \\ &= \sim (\sim P) \wedge \sim Q && \text{by DeMorgan's Laws (see \#1c above)} \\ &= P \wedge \sim Q && \text{by the definition of negation} \end{aligned}$$

5. Think up your own favorite conditional statement that you know is **false.** Call this statement R. (So, $R : P \Rightarrow Q$.)

   (a) Write R as a sentence.

   ans: If $x^2 \geq 0$, then $x \geq 0$.

   (b) Write $\boxed{\sim R}$ as a sentence using both logical structures: $\boxed{\sim (P \Rightarrow Q)}$ and $\boxed{P \wedge \sim Q}$.

   ans:

   $\sim (P \Rightarrow Q)$: It is not the case that if $x^2 \geq 0$, then $x \geq 0$.

   $P \wedge \sim Q$: It is possible for $x^2 \geq 0$ and $x < 0$.

6. What ideas/concepts/skills do you think this worksheet was supposed to teach you?

   - There are two ways to show two statements are logically equivalent: (1) a truth table or (2) an argument. If you use option (2) you need to use a rigid and carefully justified argument.
   - The facts: $\boxed{P \Rightarrow Q = \sim P \vee Q}$ and $\boxed{\sim (P \Rightarrow Q) = P \wedge \sim Q}$ are facts you should know because they get used a lot. One of the reasons is because it is sometimes easier to see (prove) a statement in its alternative form.