1. Review

   (a) $\forall n \in \mathbb{N}, 2n^2 - n \geq 1$ "For every natural number n, $2n^2 - n \geq 1$."

   - universal quantifier
   - big "and" statement

   $(2\cdot 1^2 - 1 \geq 1) \wedge (2\cdot 2^2 - 2 \geq 1) \wedge (2\cdot 3^2 - 3 \geq 1) \wedge \ldots$

   (b) $\exists n \in \mathbb{N}, 2n^2 - n > 10$ "There exists some natural number n such that $2n^2 - n \geq 10$."

   - existentially quantified statement
   - big "or" statement

   $(2\cdot 1^2 - 1 > 10) \vee (2\cdot 2^2 - 2 > 10) \vee (2\cdot 3^2 - 3 > 10) \vee \ldots$

2. For each statement below, write it using universal and/or existential quantifiers. Then determine their truth values

   (a) Every integer is a rational number. $\forall n \in \mathbb{Z}, n \in \mathbb{Q}$

   True. $\mathbb{N} \subseteq \mathbb{Q}$.

   (b) There are rational numbers whose square is rational. $\exists q \in \mathbb{Q}, q^2 \in \mathbb{Q}$

   True. Example $2 \in \mathbb{Q}$ and $2^2 = 4 \in \mathbb{Q}$.

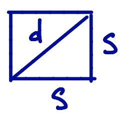
   (c) $a = \sqrt{a^2}$ for all real numbers $\forall a \in \mathbb{R}, a = \sqrt{a^2}$.

   False. Example $-1 \in \mathbb{R}$ and $-1 \neq 1 = \sqrt{(-1)^2}$.

(d) There are squares with integer values for the sides and the diagonals.

$\exists$ square with sides s and diagonal d, $s \in \mathbb{Z}$ and $d \in \mathbb{Z}$.

False.



So, $2s^2 = d^2$. So $d = \sqrt{2}\, s$. Thus, if $s \in \mathbb{Z}$, then $d \notin \mathbb{Z}$. If $d \in \mathbb{Z}$, then $s = \frac{d}{\sqrt{2}} \notin \mathbb{Z}$.

(e) Every integer that is not positive must be negative.

$\forall n \in \mathbb{Z}$, if $n \not> 0$, then $n < 0$.

False. $n = 0$ is not positive and is also not negative.

(f) For every real number a, there is some quadratic polynomial $p(x)$ where a is a root of $p(x)$.

$\forall a \in \mathbb{R}$, $\exists p(x) = bx^2 + cx + d$, $p(a) = 0$

True. Construct $p(x) = x^2 - a^2$

(g) For every quadratic polynomial $p(x)$, there is some real number a, where a is a root of $p(x)$.

$\forall p(x) = bx^2 + cx + d$, $\exists a \in \mathbb{R}$, $p(a) = 0$.

False. $p(x) = x^2 + 2$ has no real roots.

(h) If $r \in \mathbb{R}$, then $f(x) = \frac{x+r}{x^2+r^3}$ is continuous on $\mathbb{R}$. $\forall r \in \mathbb{R}$, $f(x) = \dfrac{x+r}{x^2+r^3}$ is continuous.

False. For $r = -1$, $x^2 + r^3 = x^2 - 1$. So $f(x)$ would be discontinuous at $x = \pm$.

(i) If $f : \mathbb{R} \to \mathbb{R}$ has a horizontal asymptote, then $\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$ is defined.

$\forall f : \mathbb{R} \to \mathbb{R}$, if $f(x)$ has a horizontal asymptote, then $\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$ is defined

$\forall x$, $P(x) \Rightarrow Q(x) \vee R(x)$

What we can learn from a second pass.

1. Review

(a) $\forall n \in \mathbb{N}, 2n^2 - n \geq 1$ $= (2\cdot 1^2 - 1 \geq 1) \wedge (2\cdot 2^2 - 2 \geq 1) \wedge (2\cdot 3^2 - 3 \geq 1) \wedge \dots$

$= \forall n, P(n) = R$

What is $\sim R$? How do you negate a universally quantified statement?

We know $\sim(P \wedge Q) = \sim P \vee \sim Q$. So..

$\sim(\forall n, P(n)) = \exists n, \sim P(n)$ ← Look @ $*$ ↓

(b) $\exists n \in \mathbb{N}, 2n^2 - n > 10$ $= (2\cdot 1^2 - 1 > 10) \vee (2\cdot 2^2 - 2 > 10) \vee (2\cdot 3^2 - 3 > 10) \vee \dots$

$= \exists n, P(n) = R$

What is $\sim R$?

$\sim(\exists n, P(n)) = \forall n, \sim P(n)$ ← look @ ☆ ↓

Now, we know either $\underline{\forall x, P(x)}$ or $\underline{\exists x, \sim P(x)}$ is true

Either $\underline{\exists x, P(x)}$ or $\underline{\forall x, \sim P(x)}$ is true.

(b) There are rational numbers whose square is rational. $\exists q \in \mathbb{Q}, q^2 \in \mathbb{Q}$

True. Example $2 \in \mathbb{Q}$ and $2^2 = 4 \in \mathbb{Q}$.

$*$ (c) $a = \sqrt{a^2}$ for all real numbers $\forall a \in \mathbb{R}, a = \sqrt{a^2}$.

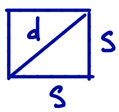
False. Example $-1 \in \mathbb{R}$ and $-1 \neq 1 = \sqrt{(-1)^2}$.

We showed: $\boxed{\exists a \in \mathbb{R}, a \neq \sqrt{a^2}}$

We showed $\forall$ square, $s \notin \mathbb{Z}$ or $d \notin \mathbb{Z}$. $\sim(P \wedge Q) = \sim P \vee \sim Q$

(d) There are squares with integer values for the sides and the diagonals.

$\exists$ square with side s and diagonal d, $s \in \mathbb{Z}$ and $d \in \mathbb{Z}$.

False. 

So, $2s^2 = d^2$. So $d = \sqrt{2}\, s$. Thus, if $s \in \mathbb{Z}$, then $d \notin \mathbb{Z}$. If $d \in \mathbb{Z}$, then $s = \frac{d}{\sqrt{2}} \notin \mathbb{Z}$.

(e) Every integer that is not positive must be negative. $\boxed{\forall n,\ P(n) \Rightarrow Q(n)}$ its negation?

$\forall n \in \mathbb{Z}$, if $n \not> 0$, then $n < 0$.

False. $n = 0$ is not positive and is also not negative.

$\boxed{\exists n, \sim(P \Rightarrow Q) = \exists n,\ P(n) \wedge \sim Q(n)}$

(f) For every real number a, there is some quadratic polynomial $p(x)$ where a is a root of $p(x)$.

$\forall a \in \mathbb{R}$, $\exists p(x) = bx^2 + cx + d$, $p(a) = 0$

True. Construct $p(x) = x^2 - a^2$

order matters $\boxed{\forall a\, \exists p(x),\ p(a) = 0} \neq \boxed{\exists p(x), \forall a,\ p(a) = 0}$

(g) For every quadratic polynomial $p(x)$, there is some real number a, where a is a root of $p(x)$.

$\forall p(x) = bx^2 + cx + d$, $\exists a \in \mathbb{R}$, $p(a) = 0$.

False. $p(x) = x^2 + 2$ has no real roots.

We showed $\exists p(x)$, $\forall a \in \mathbb{R}$, $p(a) \neq 0$.

(h) If $r \in \mathbb{R}$, then $f(x) = \frac{x+r}{x^2+r^3}$ is continuous on $\mathbb{R}$. $\forall r \in \mathbb{R}$, $f(x) = \frac{x+r}{x^2+r^3}$ is continuous.

$P \Rightarrow Q \equiv \forall$ acceptable P, Q

conditional statements can be rephrased as universally quantified statements.

(i) If $f : \mathbb{R} \to \mathbb{R}$ has a horizontal asymptote, then $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ is defined.

$\forall f : \mathbb{R} \to \mathbb{R}$, if $f(x)$ has a horizontal asymptote, then $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ is defined

$\forall x$, $P(x) \Rightarrow Q(x) \vee R(x)$