1. Review

   (a) $\sim\big(\forall n \in \mathbb{N},\, 2n^2 - n \geq 1\big)$ $= \exists n \in \mathbb{N},\ 2n^2 - n < 1$

   (b) $\sim\big(\exists n \in \mathbb{N},\, 2n^2 - n > 10\big)$ $= \forall n \in \mathbb{N},\ 2n^2 - n \leq 10$

2. From the previous sheet:

   d. There are squares with integer values for the sides and the diagonals.

$$\exists \text{ squares with side } s \text{ and diagonal } d, \big(s \in \mathbb{Z} \wedge d \in \mathbb{Z}\big)$$

   We concluded it was false. What do we need to show?

$\forall$ square, $s \notin \mathbb{Z} \vee d \notin \mathbb{Z}$.

Argument: We know $2s^2 = d^2$. If $s \in \mathbb{Z}$, then $d = \sqrt{2s^2} = \sqrt{2}\,s \notin \mathbb{Z}$.

If $d \in \mathbb{Z}$, then $s = \sqrt{\frac{d^2}{2}} = \frac{d}{\sqrt{2}} \notin \mathbb{Z}$.

   e. Every integer that is not positive must be negative.

$$\forall n \in \mathbb{N}, \sim\big(n > 0\big) \Rightarrow \big(n < 0\big)$$

   We concluded it was false. What do we need to show?

$\exists n \in \mathbb{N},\ \sim(n>0) \wedge \sim(n<0) \equiv \exists n \in \mathbb{N},\ n \leq 0 \wedge n \geq 0.$

Argument. Pick $n = 0$. ...

   g. For every quadratic polynomial $p(x)$, there is some real number a, where a is a root of $p(x)$.

$$\forall p(x) \in \mathbb{P}_2(x),\, \exists a \in \mathbb{R},\, p(a) = 0, \quad \text{where } \mathbb{P}_2(x) \text{ is the set of degree 2 polynomials}$$

   We concluded it was false. What do we need to show?

$\exists p(x),\ \forall a \in \mathbb{R},\ p(a) \neq 0.$

Argument. Pick $p(x) = x^2 + 1$. Since $\sqrt{-1} \notin \mathbb{R}$, $p(x)$ has no roots.

3. Logical Inference:

- How to know a collection of true statments implies another statement.. or NoT!
- Focus on logical structure. of an argument: valid or not valid.
- Ex: P_1, P_2, $\vdots$, P_n, —— P_{n+1}. This is a valid argument if $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \Rightarrow P_{n+1}$ for all possible input truth values.
- The argument is invalid if there are any truth values for which all hypotheses are true and conclusion is false.

4. Modus Ponens

$$\begin{array}{l} P \Rightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

- all possible truth values of input
- Check truth values of all hypotheses

P	Q	$P \Rightarrow Q$	P	Q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

- Only consider rows where both hypotheses are true
- For these rows, check that the conclusion is true ✓ Thus, it's valid.

Ex If $f(x)$ is a polynomial, then $\lim_{x \to \infty} f(x) = \infty$. Since $f(x) = \sin(x)$ is a polynomial, we can conclude that $\lim_{x \to \infty} \sin x = \infty$.

[A logically valid argument... It's the hypotheses that are the baddies.]

5. The most common fallacy (invalid argument):

$$\begin{array}{l} P \Rightarrow Q \\ Q \\ \hline \therefore P \end{array}$$

- Need to find truth values for P and Q such that
  1. $P \Rightarrow Q$ is true and
  2. Q is true but
  3. P is not true
- Ans: $Q = T$, $P = F$, Then $P \Rightarrow Q$ is $F \Rightarrow T$ which is true.

Most common Version:

NTS $A = B$.

"Proof" $A = B \Rightarrow C = D \Rightarrow E = F \Rightarrow 1 = 1$ ✓

So $A = B$.

one more comment →

You can see that this is not valid by making an argument for which all the hypotheses are true but the conclusion is incorrect.

If $f(x)$ is differentiable, then $f(x)$ is continuous.
Since $f(x)=|x|$ is continuous, we can conclude
$f(x)=|x|$ is differentiable.

Observe: Complete sentences with correct grammar.
No sentence begins with a symbol.

6. For each argument below, (a) determine whether it is valid or invalid, (b) write an argument in English that models the logical structure of the argument.

(a) $$\begin{array}{l} P \Rightarrow Q \\ \sim P \\ \hline \sim Q \end{array}$$

INVALID

Choose $P=F$, $Q=T$

Then $P \Rightarrow Q$ is $F \Rightarrow T$ which is true and $\sim P$ is true. So both hypotheses are true. But $\sim Q$ is false.

Argument: If $f(x)$ is differentiable, then $f(x)$ is continuous. Since $f(x)=|x|$ is not differentiable everywhere, we can conclude $f(x)=|x|$ is not continuous.

(b) $$\begin{array}{l} P \Rightarrow Q \\ \sim Q \\ \hline \sim P \end{array}$$

VALID

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Argument: If $x \geq 10$, then $x \geq 5$. Since $a < 5$, we can conclude $a < 10$.

(c) $$\begin{array}{l} P \vee Q \\ \sim P \\ \hline Q \end{array}$$

VALID

P	Q	$P \vee Q$	$\sim P$	Q
T	T	T	F	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

argument: All materials are in Canvas or on the webpage. The quiz is not in Canvas. Thus, the quiz is on the webpage.

7. Show that $\boxed{P \Rightarrow Q}$ is logically equivalent to $\boxed{\sim Q \Rightarrow \sim P.}$ (Note: $\sim Q \Rightarrow \sim P$ is called the **contrapositive** of $P \Rightarrow Q$.)

8. Rewrite each theorem below with its equivalent contrapositive statement. Note that the "Let..." sentence does not change.

   (a) If two sides of a triangle are congruent (aka of equal length), then the two angles opposite those sides are congruent (aka are equal in measure).

   If the angles opposite two sides of a triangle are unequal, then the sides are not congruent.

   (b) Let $f(x)$ be defined on the interval $[a,b]$. If $f(x)$ is continuous on $[a,b]$, then for every y-value, y_0, strictly between $f(a)$ and $f(b)$ there exists an x-value, x_0, in (a,b) such that $f(x_0) = y_0$.

   Let $f(x)$ be defined on $[a,b]$.
   If there exists a y_0 strictly between $f(a)$ and $f(b)$ such that for every $x_0 \in (a,b)$, $f(x_0) \neq y_0$, then $f(x)$ is not continuous on $[a,b]$.