Solutions

1. Definitions and Facts

   (a) An integer n is **even** if $\exists k \in \mathbb{Z}$, $n = 2k$.

   (b) An integer n is **odd** if $\exists k \in \mathbb{Z}$, $n = 2k+1$.

   (c) Let $a, b \in \mathbb{Z}$. We say a **divides** b if $\exists k \in \mathbb{Z}$, $ak = b$.

   Alternate wording: a **is a divisor of** b OR b is **a multiple of** a

   (d) A number $n \in \mathbb{N}$ is **prime** if it has exactly two distinct divisors.

   A number $n \in \mathbb{N}$ is **composite** if it has more than two distinct divisors.

   (e) Let $a, b \in \mathbb{Z}$. The **greatest common division of** a **and** b (denoted $\gcd(a, b)$) is the largest integer n such that $n|a$ and $n|b$.

   (f) $a, b \in \mathbb{Z} - \{0\}$. The **least common multiple of** a **and** b (denoted $\text{lcm}(a, b)$) is the smallest positive integer n such that $a|n$ and $b|n$

   (g) **Fact 4.1:** If $a, b, \in \mathbb{Z}$, then $a+b$, $a-b$, and ab are also in $\mathbb{Z}$.

   Alternate wording: The integers are closed under addition and multiplication.

   (h) **The Division Algorithm** For every $a \in \mathbb{Z}$ and $b \in \mathbb{N} - \{0\}$, there exists unique integers q and r such that

   $$a = qb + r, \quad \text{where} \quad 0 \leq r < b.$$

2. Outline for a **Direct Proof**

> **Proposition:** If P, then Q.
> **Proof:** (direct) Suppose P (is true).
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> Thus, Q (is true). □

3. Prove that for every integer m, if n is even, then $3n^2 - 5mn - 8$ is also even.

   Proof. (direct) Let $m \in \mathbb{Z}$ and suppose n is even. Then, by the definition of even, there exists an integer k such that $n = 2k$. Let $\ell = 3k^2 - 5km - 4$.
   Now,

$$
\begin{aligned}
3n^2 - 5mn - 8 &= 3(2k)^2 - 5(2k)m - 8 && \text{by substituting } n = 2k \\
&= 6k^2 - 10km - 8 && \text{by rules of multiplication} \\
&= 2(3k^2 - 5km - 4) && \text{by factoring out a 2} \\
&= 2\ell && \text{by substituting } \ell = 3k^2 - 5km - 4.
\end{aligned}
$$

   By Fact 4.1, since k and m are integers, we know $3k^2 - 5km - 4$ is also an integer. Thus, $3n^2 - 5mn - 8 = 2\ell$, where $\ell \in \mathbb{Z}$. Thus, $3n^2 - 5mn - 8$ is even by definition. □

4. Let $x, y \in \mathbb{R}^+$. Prove that if $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.

   Q: Do you think all of the hypotheses are needed?

   Proof. (direct) Let $x, y \in \mathbb{R}^+$ such that $x \leq y$. By subtracting x from both sides, we obtain $0 \leq y - x$. Since x and y are both positive, we know that $\sqrt{x}$ and $\sqrt{y}$ are defined. Thus, we can factor $y - x$ as a difference of squares to get $y - x = (\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x})$.
   Using $0 \leq y - x$ and $y - x = (\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x})$, we conclude $0 \leq (\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x})$.
   Since $\sqrt{y} + \sqrt{x} > 0$, we can divide $0 \leq (\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x})$ by $\sqrt{y} + \sqrt{x}$ to obtain $0 \leq \sqrt{y} - \sqrt{x}$. By adding $\sqrt{2}$ to both sides of the previous inequality, we obtain $\sqrt{x} \leq \sqrt{y}$, which is what we wanted to prove. □

   Q: Did we use all the hypotheses?

5. Rigid and Unforgiving Rules

   (a) All parts of all proofs are complete sentences which begin with a word in English and end with a period. No sentence fragments.

   (b) The following symbols never appear: $\forall$, $\exists$, $\Rightarrow$, $\vee$, $\wedge$.

   (c) **All** strings of equalities are aligned vertically, with justification.

   (d) Don't use a fact if you haven't proved it.

6. Let $a, b \in \mathbb{Z}$. Prove that if $a \big| b$, then $a^2 \big| b^2$.

Proof. (direct) Let $a, b \in \mathbb{Z}$ such that $a \big| b$. Then, by the definition of **divides**, there is an integer k such that $ak = b$. Let $\ell = k^2$. Squaring both sides of the previous equation, we obtain

$$\begin{aligned} b^2 &= (ak)^2 & \\ &= a^2(k^2) & \text{factoring out } a^2 \\ &= a^2\ell & \text{by substituting} \ell = k^2. \end{aligned}$$

Since $b^2 = a^2\ell$ for $\ell \in \mathbb{Z}$, $a^2 \big| b^2$ by the definition of **divides**.

□

7. Let $x, y \in \mathbb{R}^+$. Prove that $2\sqrt{xy} \leq x + y$.

Proof. (direct) Let $x, y \in \mathbb{R}^+$. Since x and y are real numbers, we know

$$0 \leq (x - y)^2 = x^2 - 2xy + y^2.$$

If we add $4xy$ to both ends of the inequality above, we obtain

$$4xy \leq x^2 + 2xy + y^2.$$

Factoring the right-hand side yields

$$4xy \leq (x + y)^2.$$

Since x and y are both positive real numbers, we know that both $4xy$ and $(x + y)^2$ are positive. Thus, we can apply the result from #4 above and use the fact that $4xy \leq (x + y)^2$ to conclude that $2\sqrt{xy} \leq x + y$, which is what we wanted to show.

□

Start with what you know about $(x - y)^2$.