1. For every $a,b,c \in \mathbb{N}$, $\text{lcm}(ca,cb) = c \cdot \text{lcm}(a,b)$.

Proof. Let $a,b,c \in \mathbb{N}$, $m = \text{lcm}(ca,cb)$, and $n = c \cdot \text{lcm}(a,b)$. To show that $m = n$, we will show that $m \leq n$ and $n \leq m$.

Since $m = \text{lcm}(ca,cb)$, by the definition of least common multiple, we know there exist integers k_1 and k_2 such that
$$cak_1 = m = cbk_2.$$

Since $c \neq 0$, we can divide each equation above to obtain
$$ak_1 = \frac{m}{c} = bk_2,$$
where we know k_1, k_2, and $\frac{m}{c}$ are all integers. Thus, we have shown that $\frac{m}{c}$ is a common multiple of a and b. By the definition of least common multiple, we know $\text{lcm}(a,b) \leq \frac{m}{c}$. Multiplying the previous inequality by c gives
$$n = c \cdot \text{lcm}(a,b) \leq c \cdot \frac{m}{c} = m,$$
and we conclude that $n \leq m$.

To show the reverse inequality, we apply the definition of least common multiple to $\text{lcm}(a,b)$ to conclude that there exist integers k_1 and k_2 such that
$$ak_1 = \text{lcm}(a,b) = bk_2.$$

Multiplying both equations by c, we obtain
$$cak_1 = c \cdot \text{lcm}(a,b) = cbk_2.$$

Thus, we have shown that $c \cdot \text{lcm}(a,b)$ is a common multiple of both ca and cb. Thus,
$$m = \text{lcm}(ca,cb) \leq c \cdot \text{lcm}(a,b) = n.$$

□

2. Every multiple of 4 can be written in the form $1 + (-1)^n(2n-1)$ for some $n \in \mathbb{N}$.

Proof. Let $m = 4a$ where $a \in \mathbb{Z}$. We will proceed by cases based on the value of a.

Case 1: Suppose $a = 0$.

Choose $n = 1$. Observe $1 \in \mathbb{N}$. Now, $1 + (-1)^n(2n-1) = 1 - 1 = 0 = 4 \cdot 0$, which is what we needed to show.

Case 2: Suppose $a > 0$.

Choose $n = 2a$. Observe that since $a \in \mathbb{N}$, we know $n = 2a \in \mathbb{N}$. Now, $1-(-1)^n(2n-1) = 1+(4a-1) = 4a$, which is what we needed to show.

Case 3: Suppose $a < 0$.

Choose $n = -2a+1$. Observe that since a is a negative integer, $2a+1 \in \mathbb{N}$. Now, $1-(-1)^n(2n-1) = 1-(2(-2a+1)-1) = 1-(-4a+2-1) = 4a$, which is what we needed to show.

□

3. For every integer n, n^2+3n+3 is odd.

Proof. We will proceed by cases depending on the parity of n.

Case 1: Suppose n is even.

By definition of even, there is an integer k such that $n = 2k$. Thus,

$$n^2+3n+3 = (2k)^2+3(2k)+3 = 2(2k^2+3k+1)+1.$$

Since k is an integer, Fact 4.1 implies that $\ell = 2k^2+3k+1$ is also an integer. Thus, we have shown that when n is even, $n^2+3n+3 = 2\ell+1$, where $\ell \in \mathbb{Z}$. Thus, n^2+3n+3 is even by definition in this case.

Case 2: Suppose n is odd.

By definition of odd, there is an integer k such that $n = 2k+1$. Thus,

$$n^2+3n+3 = (2k+1)^2+3(2k+1)+3 = 2(2k^2+5k+3)+1.$$

Since k is an integer, Fact 4.1 implies that $\ell = 2k^2+5k+3$ is also an integer. Thus, we have shown that when n is odd, $n^2+3n+3 = 2\ell+1$, where $\ell \in \mathbb{Z}$. Thus, $3n^2+4n+6$ is even by definition in this case.

□

4. Let $a, b \in \mathbb{N}$. If $\gcd(a,b) > 1$, then $b \big| a$ or b is not prime.

   Proof. Let $a, b \in \mathbb{N}$ such that $\gcd(a,b) > 1$. We will proceed by cases based on whether or not b is prime.

   Case 1: b is not prime.

   Then the result follows immediately.

   Case 2: b is prime.

   Since b is prime, its only divisors are 1 and b. Thus, $\gcd(a,b) \in \{1, b\}$. But $\gcd(a,b) > 1$. Thus, $\gcd(a,b) \neq 1$ and therefore $\gcd(a,b) = b$. Since $\gcd(a,b) \big| a$ and $\gcd(a,b) = b$, it follows that $b \big| a$. □