<u>Theorem</u>: Let A be a square matrix.
If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are eigenvectors associated with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent.

<u>Pf</u>: Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly <u>DE</u>pendent.

Let k be the smallest index such that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ is linearly dependent.

$$* \quad \boxed{\vec{v}_k = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_{k-1} \vec{v}_{k-1},} \quad c_i \in \mathbb{R},$$

and $c_1, c_2, c_3, \dots, c_{k-1}$ cannot <u>all</u> be zero, since $\vec{v}_1 \neq \vec{0}$.

Now, apply A to both sides of $*$

$$A\vec{v}_k = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \dots + c_{k-1} A\vec{v}_{k-1}$$

Use the fact that they are eigenvectors

$$\lambda_k \vec{v}_k = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + \dots + c_{k-1} \lambda_{k-1} \vec{v}_{k-1}$$

Trick: Multiply $*$ by λ_1

$$\lambda_k \vec{v}_k = c_1 \lambda_k \vec{v}_1 + c_2 \lambda_k \vec{v}_2 + \dots + c_{k-1} \lambda_k \vec{v}_{k-1}$$

Subtract bottom from top

$$\vec{0} = c_1(\lambda_1 - \lambda_k)\vec{v}_1 + c_2(\lambda_2 - \lambda_k)\vec{v}_2 + \dots + c_{k-1}(\lambda_{k-1} - \lambda_k)\vec{v}_{k-1}$$

But $v_1, v_2, \dots, v_{k-1}$ are linearly <u>independent</u>. But c_i's are not all zero, <u>and</u>, $\lambda_i - \lambda_k$ cannot be zero. So we have a contradiction.