Math 314 # Midterm 1 **Fall 2022**

Name: ____________________

Rules:

You have one hour to complete the exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten sheet of notes.

Except for problem 1, you may use technology to find the reduced echelon form of a matrix.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	15	
2	14	
3	18	
4	18	
5	10	
6	10	
7	15	
Extra Credit	5	
Total	100	

1. (15 points) Use Gauss-Jordan reduction to find the reduced echelon form of the matrix A below. You must show your work and state the row operations you are performing.

$$A = \begin{pmatrix} 2 & -2 & 2 & 8 \\ 1 & 0 & 2 & 6 \\ 0 & 2 & 0 & 1 \end{pmatrix} \quad \tfrac{1}{2}r_1 \to r_1 \quad \begin{pmatrix} 1 & -1 & 1 & 4 \\ 1 & 0 & 2 & 6 \\ 0 & 2 & 0 & 1 \end{pmatrix} \quad r_2 - r_1 \mapsto r_2 \quad \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} r_1 + r_2 \to r_1 \\ r_3 - 2r_2 \to r_3 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -3 \end{pmatrix} \quad \begin{matrix} r_1 + r_3 \mapsto r_1 \\ r_2 + \tfrac{1}{2}r_3 \mapsto r_2 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & \tfrac{1}{2} \\ 0 & 0 & -2 & -3 \end{pmatrix}$$

$$-\tfrac{1}{2}r_3 \to r_3 \quad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & \tfrac{1}{2} \\ 0 & 0 & 1 & \tfrac{3}{3} \end{pmatrix}$$

2. (14 points) Let $T = \{a_0 + a_1x + a_2x^2 + a_3x^3 \ : \ a_0 + a_1 = a_3 \text{ and } a_2 = 0\}$ be a subset of, $\mathcal{P}_3$, the vector space of all polynomials of degree 3 or less.

   Is T a **subspace** of $\mathcal{P}_3$? Justify your answer.

   Note that a complete answer is not simply a computation (or computations) but includes and explanation in words indicating how that computation is used to derive your conclusion.

① Way 1 Answer: Yes. Justification: Show $T = \text{span}(S)$ for appropriate S.

$$T = \{a_0 + a_1x + (a_0+a_1)x^3 : a_0, a_1 \in \mathbb{R}\} = \{a_0(1+x^3) + a_1(x+x^3) : a_0, a_1 \in \mathbb{R}\}$$

$= \text{span}(\{1+x^3, x+x^3\})$. Since T can be written as a span of the set $\{1+x^3, x+x^3\}$ it must be a subspace.

② way2: Show T is closed under $+$ and scalar $\cdot$

So $T = \{a_0 + a_1x + (a_0+a_1)x^3 : a_0, a_1 \in \mathbb{R}\}$.

- Let $a_0+a_1x+(a_0+a_1)x^3$, $b_0+b_1x+(b_0+b_1)x^3$ be in T.

  Then $(a_0+a_1x+(a_0+a_1)x^3)+(b_0+b_1x+(b_0+b_1)x^3)$

  $= (a_0+b_0)+(a_1+b_1)x+((a_0+b_0)+(a_1+b_1))x^3$ is also in T.

- Let $r \in \mathbb{R}$.

  Then $r(a_0+a_1x+(a_0+a_1)x^3) = ra_0 + ra_1x + (ra_0+ra_1)x^3$ is also in T.

3. (18 points) Let $A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & -2 & 0 & 0 \\ 1 & 3 & -2 & 1 & -1 \end{pmatrix}$.

   (a) Find a basis for the column space of A.

$$A^T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & -2 \\ 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis: $\left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

   (b) What is the rank of the matrix A. 3

   (c) Give an example of a vector $\vec{v}$ that is not in the column space of A and demonstrate that your example is correct.

Claim $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is not in column space of A.

Claim: No c_1, c_2, c_3 so that $c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

OR $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$. So the system has no solution

4. (18 points) Consider the system of linear equations below.

$$\begin{cases} w + x + y + z = 5 \\ -w + x - y + z = 5 \\ w + x \quad\quad - z = 1 \end{cases}$$

(a) Solve the system of linear equations and express your answer in vector form.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ -1 & 1 & -1 & 1 & 5 \\ 1 & 1 & 0 & -1 & 1 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right). \text{ So } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2w-4 \\ -w+5 \\ -2w+4 \\ w \end{pmatrix}$$

$$W = \left\{ \begin{pmatrix} -4 \\ 5 \\ 4 \\ 0 \end{pmatrix} + w \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix} : w \in \mathbb{R} \right\}$$

(b) In your answer above, identify the particular solution and homogeneous solution.

particular : $\vec{p} = \begin{pmatrix} -4 \\ 5 \\ 4 \\ 0 \end{pmatrix}$, homogeneous $\vec{h} = \left\{ w \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix} : w \in \mathbb{R} \right\}$

(c) Show that the solution set you found in part (a) is **not** a vector space under the standard vector addition and scalar multiplication in $\mathbb{R}^4$.

1 way 1 : $\vec{0} \notin W$. (no zero vector.) The only way to make 4th coordinate zero, is to choos $w=0$. But if $w=0$, then none of the other coordinates are zero.

5. (10 points) Do the vectors $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$ span the space $\mathcal{M}_{2\times2}$? Justify your answer.

We must find c_1, c_2, c_3, c_4, so that

$$c_1\begin{pmatrix}1&0\\0&1\end{pmatrix}+c_2\begin{pmatrix}0&1\\1&0\end{pmatrix}+c_3\begin{pmatrix}1&1\\0&1\end{pmatrix}+c_4\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$\boxed{\text{(way 1)}}$

So $\left(\begin{array}{cccc|c}1&0&1&0&a\\0&1&1&0&b\\0&1&0&1&c\\1&0&1&0&d\end{array}\right)\xrightarrow{\text{rref}}\left(\begin{array}{cccc|c}1&0&0&1&*\\0&1&0&1&*\\0&0&1&-1&*\\0&0&0&0&*\end{array}\right)$.

No. Since the coefficient matrix has rank 3, the dimension of the column space is 3. But $M_{2\times2}$ has dimension 4.

$\boxed{\text{way 2}}$ chosen random

$\left(\begin{array}{cccc|c}1&0&1&0&1\\0&1&1&0&2\\0&1&0&1&3\\1&0&1&0&4\end{array}\right)\xrightarrow{\text{rref}}\left(\begin{array}{cccc|c}1&0&0&1&0\\0&1&0&1&0\\0&0&1&-1&0\\0&0&0&0&1\end{array}\right)$. So we have shown the vectors do not span $\begin{pmatrix}1&2\\3&4\end{pmatrix}$.

6. (10 points) The vectors $B=\langle 1, 1+x, 2x^2\rangle$ form a basis for $\mathcal{P}_2$ the vector space of all polynomials of degree 2 or less. Write the representation of the vector $2-4x+5x^2$ in terms of the basis B.

Way 1: by inspection.

$$\text{rep}_B(\vec{v})=\begin{pmatrix}6\\-4\\5/2\end{pmatrix}$$

check

$6(1)-4(1+x)+\frac{5}{2}(2x^2)$

$=6-4-4x+5x^2$ ✓

Way 2: systematically:

we want $c_1(1)+c_2(1+x)+c_3(2x^2)=2-4x+5x^2$ or

$(c_1+c_2)+c_2x+2c_3x^2=2-4x+5x^2$?

So: $c_1+c_2=2$

$c_2=-4$

$2c_3=5$.

Solve by inspection or by

$$\left[\begin{array}{ccc|c}1&1&0&2\\0&1&0&-4\\0&0&2&5\end{array}\right]\xrightarrow{\text{rref}}\left[\begin{array}{ccc|c}1&0&0&6\\0&1&0&-4\\0&0&1&5/2\end{array}\right].$$

7. (15 points) Let $S = \{\vec{s_1}, \vec{s_2}, \vec{s_3}, \cdots \vec{s_{10}}\}$ be a subset of vectors from the vector space V. Assume that the the linear equation

$$c_1\vec{s_1} + c_2\vec{s_2} + c_3\vec{s_3} + \cdots + c_{10}\vec{s_{10}} = \vec{0}$$

has $c_1 = 2,\ c_2 = 5,\ c_3 = -4,\ c_4 = c_5 = \cdots = c_{10} = 0$ as a solution.

(a) Can you conclude that S is linearly dependent or linearly independent? Why or why not? Explain your reasoning.

S is linearly dependent because there exists a nontrivial solution to the equation.

(b) What can you conclude about $[S - \{\vec{s_1}\}]$ and $[S]$? Are they equal? Is one strictly smaller than the other? Do you have enough information to draw a conclusion? Explain your reasoning.

$[S - s_1] = [S]$. Since $\vec{s_1}$ can be written as a linear combination of $S - \vec{s_1}$, we know $s_1 \in [S]$. So $[S - \vec{s_1}] = [S]$.

(c) Is the given solution (namely that $c_1 = 2,\ c_2 = 5,\ c_3 = -4,\ c_4 = c_5 = \cdots = c_{10} = 0$) **unique**? Do you have enough information to draw a conclusion? Explain your reasoning.

No. $c_1 = c_2 = \dots = c_{10} = 0$ is also a solution.

Extra Credit: (5 points) Determine whether the vectors $f(x) = x^2,\ g(x) = 2^x$, and $h(x) = 3^x$ are linearly independent in the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Justify your answer.

Extra Credit: (5 points) Determine whether the vectors $f(x) = x^2$, $g(x) = 2^x$, and $h(x) = 3^x$ are linearly independent in the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Justify your answer.

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Claim: linearly independent.

We need to show that $c_1 x^2 + c_2 2^x + c_3 3^x = 0$ has only the solution $c_1 = c_2 = c_3 = 0$.

If $x = 0$, then $c_2 + c_3 = 0$

If $x = 1$, then $c_1 + 2c_2 + 3c_3 = 0$

If $x = 2$, then $4c_1 + 4c_2 + 9c_3 = 0$.

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 4 & 4 & 9 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, $c_1 = c_2 = c_3$ is the unique solution.