Math 314 Midterm 1 Fall 2022

Name: ____________________

Rules:

You have one hour to complete the exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten sheet of notes.

Except for problem 1, you may use technology to find the reduced echelon form of a matrix.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	15	
2	14	
3	18	
4	18	
5	10	
6	10	
7	15	
Extra Credit	5	
Total	100	

1. (15 points) Use Gauss-Jordan reduction to find the reduced echelon form of the matrix A below. You must show your work and state the row operations you are performing.

$$A = \begin{pmatrix} 2 & -2 & 2 & 8 \\ 1 & 0 & 2 & 6 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

2. (14 points) Let $T = \left\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ : \ a_0 + a_1 = a_3 \text{ and } a_2 = 0\right\}$ be a subset of, $\mathcal{P}_3$, the vector space of all polynomials of degree 3 or less.

   Is T a **subspace** of $\mathcal{P}_3$? Justify your answer.

   Note that a complete answer is not simply a computation (or computations) but includes and explanation in words indicating how that computation is used to derive your conclusion.

3. (18 points) Let $A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & -2 & 0 & 0 \\ 1 & 3 & -2 & 1 & -1 \end{pmatrix}$.

   (a) Find a basis for the column space of A.

   (b) What is the rank of the matrix A.

   (c) Give an example of a vector $\vec{v}$ that is not in the column space of A and demonstrate that your example is correct.

4. (18 points) Consider the system of linear equations below.

$$\begin{cases} w + x + y + z = 5 \\ -w + x - y + z = 5 \\ w + x \quad\quad - z = 1 \end{cases}$$

(a) Solve the system of linear equations and express your answer in vector form.

(b) In your answer above, identify a particular solution and the solution set of the homogeneous system.

(c) Show that the solution set you found in part (a) is **not** a vector space under the standard vector addition and scalar multiplication in $\mathbb{R}^4$.

5. (10 points) Do the vectors $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ span the space $\mathcal{M}_{2\times 2}$? Justify your answer.

6. (10 points) The vectors $B = \langle 1, 1 + x, 2x^2 \rangle$ form a basis for $\mathcal{P}_2$ the vector space of all polynomials of degree 2 or less. Write the representation of the vector $2 - 4x + 5x^2$ in terms of the basis B.

7. (15 points) Let $S = \{\vec{s_1}, \vec{s_2}, \vec{s_3}, \cdots \vec{s_{10}}\}$ be a subset of vectors from the vector space V. Assume that the the linear equation

$$c_1\vec{s_1} + c_2\vec{s_2} + c_3\vec{s_3} + \cdots + c_{10}\vec{s_{10}} = \vec{0}$$

has $c_1 = 2,\ c_2 = 5,\ c_3 = -4,\ c_4 = c_5 = \cdots = c_{10} = 0$ as a solution.

(a) Can you conclude that S is linearly dependent or linearly independent? Why or why not? Explain your reasoning.

(b) What can you conclude about $[S - \{\vec{s_1}\}]$ and $[S]$? Are they equal? Is one strictly smaller than the other? Do you have enough information to draw a conclusion? Explain your reasoning.

(c) Is it possible to determine how many solutions to the linear equation exist? Explain your reasoning.

Extra Credit: (5 points) Determine whether the vectors $f(x) = x^2$, $g(x) = 2^x$, and $h(x) = 3^x$ are linearly independent in the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Justify your answer.