Math 314 # Midterm 1 **Fall 2022**

Name: Solutions

Rules:

You have one hour to complete the exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten sheet of notes.

Except for problem 1, you may use technology to find the reduced echelon form of a matrix or to multiply matrices.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	25	
4	20	
5	20	
6	15	
Extra Credit	5	
Total	100	

1. (10 points) Find the product AB for the matrices below, **by hand**.

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 5 & 1 \\ -3 & 0 & 0 & 1 \end{pmatrix}$$

$$(2 \times 3) \quad (3 \times 4) \quad = 2 \times 4$$

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 5 & 1 \\ -3 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 0-4+0 & -2-10+0 & 0-2+0 \\ 0+0-12 & 0-2+0 & 0-5+0 & 0-1+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -4 & -12 & -2 \\ -12 & -2 & -5 & 3 \end{pmatrix}$$

2. (10 points) Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & -1 & 4 \end{pmatrix}$. **Demonstrate** how to find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1/2 & 1/2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1/4 & -1/4 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \\ 1 & -1/4 & -1/4 \end{bmatrix}$

3. (25 points) Let $h : \mathcal{P}_2 \to \mathbb{R}^3$ by $h(ax^2 + bx + c) = \begin{pmatrix} a \\ a+c \\ 2c \end{pmatrix}$ is a linear map.

(a) Show that h respects scalar multiplication.

Need to show that for every real number r and every vector $\vec{v}$, $h(r\vec{v}) = r\,h(\vec{v})$.

$$h(r(ax^2+bx+c)) = h(rax^2+rbx+rc) = \begin{pmatrix} ra \\ ra+rc \\ 2rc \end{pmatrix}$$

$$= r\begin{pmatrix} a \\ a+c \\ 2c \end{pmatrix} = r\,h(ax^2+bx+c).$$

+5

(b) Find the image of $x^2 + 4x - 2$ under h.

$$h(x^2+4x-2) = \begin{pmatrix} 1 \\ 1-2 \\ 2(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$$

+4

(c) Is $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ in the range of h? Justify your answer.

No. If $h(ax^2+bx+c) = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$

+4 then $\begin{array}{l} a=1 \\ a+c=3 \\ 2c=6. \end{array}$. So $a=1, c=3$ but $a+c \neq 3$.

... continued on next page ....

... continued from the previous page ....

Recall that $h : \mathcal{P}_2 \to \mathbb{R}^3$ by $h(ax^2 + bx + c) = \begin{pmatrix} a \\ a+c \\ 2c \end{pmatrix}$.

(d) Determine the null space of h.

Find all ax^2+bx+c so that $\begin{pmatrix} a \\ a+c \\ 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. So $a=c=0$.

+5

Answer: null space $= \{ bx : b \in \mathbb{R} \}$

(e) Determine the rank of h by finding a basis for the range space of h.

$\operatorname{rank} h = \dim(\operatorname{range} h)$.

$\operatorname{range} h = \left\{ \begin{pmatrix} a \\ a+c \\ 2c \end{pmatrix} : a,c \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} : a,c \in \mathbb{R} \right\}$

+5

$= \operatorname{span}\left(\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\} \right)$. So, rank $= 2$

(f) Is h an isomorphism? Explain.

No. nullity of $h = 1 \neq 0$. So h is not 1-1.

+2

(OR: rank $h = 2 \neq 3$, So h is not onto.)

4. (20 points) Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map with matrix representation $H = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ (with respect to $\mathcal{E}_3$ in both domain and codomain)

It is a fact that $H^{-1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$.

(a) Find $h\left(\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}\right)$ $= \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2-3+2 \\ 3-2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$

+4

(b) Find the inverse image of $\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$ or state that it does not exist.

+4 $H^{-1}\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2-1+0 \\ -2+3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

(c) Explain why the information given in the problem implies that the nullity of h is zero.

+4 H is invertible. So H is 1-1. So nullity of $h = 0$.

+4

(d) Find the matrix representation of the linear map $h \circ h$.

$$H^2 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{pmatrix}$$

(e) Is $h \circ h$ an isomorphism? Justify your answer.

Yes.

+4 h has an inverse. So h is an isomorphism.
Thus $(h \circ h)$ has inverse $h^{-1} \circ h^{-1}$. Or alternately, H^2 has inverse $(H^{-1})^2$. In either case, $h \circ h$ has an inverse implies it is an isomorphism.

[There are many possible justifications here.]

5. (20 points) Let $B = \left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\rangle$ and let $D = \left\langle \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\rangle$ be two bases of $\mathbb{R}^2$.

(a) Write the vector $\vec{v} = \begin{pmatrix}2\\-8\end{pmatrix}_{\mathcal{E}_2}$ with respect to basis B.

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\-8\end{pmatrix}$$

x7

$$\left[\begin{array}{cc|c}1 & -1 & 2\\ 1 & 1 & -8\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c}1 & 0 & -3\\ 0 & 1 & -5\end{array}\right].$$

So $\vec{v} = \begin{pmatrix}-3\\-5\end{pmatrix}_B$

(b) Find $Rep_{B,D}(id)$, the change of basis matrix from basis B to basis D.

x7

$$\left(\begin{array}{cc|cc}2 & 1 & 1 & -1\\ 0 & 2 & 1 & 1\end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|cc}1 & 0 & 1/4 & -3/4\\ 0 & 1 & 1/2 & 1/2\end{array}\right)$$

So $Rep_{B,D}(id) = \begin{pmatrix}\frac{1}{4} & \frac{-3}{4}\\ \frac{1}{2} & 1/2\end{pmatrix}$

(c) Use your answer in part (b) to find $rep_D(\vec{v})$.

x6

$$\begin{pmatrix}\frac{1}{4} & \frac{-3}{4}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}-3\\-5\end{pmatrix} = \begin{pmatrix}-\frac{3}{4} + \frac{15}{4}\\ -\frac{3}{2} - \frac{5}{2}\end{pmatrix} = \begin{pmatrix}3\\-4\end{pmatrix}$$

6. (15 points) Short Answer

(a) Let M be a 5×7 matrix with rank 3 that represents a linear transformation h from vector space V to vector space W. Fill in the blanks:

+4

- The dimension of V is 7
- The dimension of W is 5
- The dimension of the range of h is 3
- The dimension of the null space of h is 4

(b) Assume $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map such that $\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}2\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}-1\\0\end{pmatrix}$. Find the image of the vector $\begin{pmatrix}3\\-1\end{pmatrix}$.

+3 $h\left(\begin{pmatrix}3\\-1\end{pmatrix}\right) = h\left(3\begin{pmatrix}1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\1\end{pmatrix}\right) = 3\begin{pmatrix}2\\2\end{pmatrix} - \begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}6+1\\6\end{pmatrix} = \begin{pmatrix}7\\6\end{pmatrix}$

(c) Assume A is a singular $n \times n$ matrix that represents a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Can there exist a vector $\vec{w}$ in the codomain that fails to be the image of any vector in the domain? Explain your reasoning.

+4

Since A is singular, rank $A < n$. So A represents a map this is NOT onto. So there must be some vector $\vec{v}$ in the codomain that is not the image of any vector in the domain.

(d) Assume $h : V \to W$ is a one-to-one linear transformation from vector space V to vector space W and has matrix representation H. Can you conclude that the columns of H are linearly independent? Explain.

+4

Since h is 1-1, dim(V) = dim(range of h) = rank(H). So the reduced row echelon form of H must have a leading 1 in every column. So the columns must be linearly independent.

Extra Credit (5 points) In problem 4, the matrix $H = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ was the representation of a linear transformation h. Find the matrix representation of h with respect to the basis $B = \langle \vec{e_2}, \vec{e_3}, \vec{e_1} \rangle$ (i.e. a re-ordering of the standard basis).

$$\mathrm{Rep}_{B,\mathcal{E}_3} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = M_1$$

$$\mathrm{Rep}_{\mathcal{E}_3 B} = M_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

So $\mathrm{Rep}_B(h) = M_1^{-1} \cdot H \cdot M_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix}$