Math 314 Midterm 1 Fall 2022

Name: ____________________

Rules:

You have one hour to complete the exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten sheet of notes.

Except for problem 1, you may use technology to find the reduced echelon form of a matrix or to multiply matrices.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	25	
4	20	
5	20	
6	15	
Extra Credit	5	
Total	100	

1. (10 points) Find the product AB for the matrices below, **by hand**.

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 5 & 1 \\ -3 & 0 & 0 & 1 \end{pmatrix}$$

2. (10 points) Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & -1 & 4 \end{pmatrix}$. **Demonstrate** how 10 find A^{-1}.

3. (25 points) Let $h : \mathcal{P}_2 \to \mathbb{R}^3$ by $h(ax^2 + bx + c) = \begin{pmatrix} a \\ a + c \\ 2c \end{pmatrix}$ is a linear map.

   (a) Show that h respects scalar multiplication.

   (b) Find the image of $x^2 + 4x - 2$ under h.

   (c) Is $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ in the range of h? Justify your answer.

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Recall that $h : \mathcal{P}_2 \to \mathbb{R}^3$ by $h(ax^2 + bx + c) = \begin{pmatrix} a \\ a + c \\ 2c \end{pmatrix}$.

(d) Determine the null space of h.

(e) Determine the rank of h by finding a basis for the range space of h.

(f) Is h an isomorphism? Explain.

4. (20 points) Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map with matrix representation $H = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ (with respect to $\mathcal{E}_3$ in both domain and codomain)

   It is a fact that $H^{-1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$.

   (a) Find $h\left(\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}\right)$

   (b) Find the inverse image of $\begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$ or state that it does not exist.

   (c) Explain why the information given in the problem implies that the nullity of h is zero.

   (d) Find the matrix representation of the linear map $h \circ h$.

   (e) Is $h \circ h$ an isomorphism? Justify your answer.

5. (20 points) Let $B = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle$ and let $D = \left\langle \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$ be two bases of $\mathbb{R}^2$.

   (a) Write the vector $\vec{v} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}_{\mathcal{E}_2}$ with respect to basis B.

   (b) Find $Rep_{B,D}(id)$, the change of basis matrix from basis B to basis D.

   (c) Use your answer in part (b) to find $rep_D(\vec{v})$.

6. (15 points) Short Answer

   (a) Let M be a 5×7 matrix with rank 3 that represents a linear transformation h from vector space V to vector space W. Fill in the blanks:

   - The dimension of V is ____________
   - The dimension of W is ____________
   - The dimension of the range of h is ____________
   - The dimension of the null space of h is ____________

   (b) Assume $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map such that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Find the image of the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

   (c) Assume A is a singular $n \times n$ matrix that represents a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Can there exist a vector $\vec{w}$ in the codomain that fails to be the image of any vector in the domain? Explain your reasoning.

   (d) Assume $h : V \to W$ is a one-to-one linear transformation from vector space V to vector space W and has matrix representation H. Can you conclude that the columns of H are linearly independent? Explain.

Extra Credit (5 points) In problem 4, the matrix $H = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ was the representation of a linear transformation h. Find the matrix representation of h with respect to the basis $B = \langle \vec{e_2}, \vec{e_3}, \vec{e_1} \rangle$ (i.e. a re-ordering of the standard basis).