Properties of the Determinant

1. **Theorem 1:** The determinant of an $n \times n$ matrix A can be computed by cofactor expansion across any row or down any column where the signs of the cofactors are determined by the figure below:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & & \ddots \end{bmatrix}$$

2. **Theorem 2:** The determinant of a triangular matrix is the product of the entries on the main diagonal.

3. **Theorem 3:** (How Row Operations Change the Determinant) Let A be a square matrix.
   - If matrix B is obtained by **adding a multiple of one row of A to another row of A,** then, $det(B) = det(A)$.
   - If matrix B is obtained by **interchanging two rows of A,** then, $det(B) = -det(A)$.
   - If matrix B is obtained by **multiplying a rows of A by the constant k,** then, $det(B) = k \cdot det(A)$.

4. **Theorem 4:** A square matrix A is invertible if and only if $det(A) \neq 0$.

5. **Theorem 5:** If A is a square $n \times n$ matrix, then $det(A) = det(A^T)$.

6. **Theorem 6:** If A and B are square matrices, then $det(AB) = det(A) \cdot det(B)$.

Examples of Properties of the Determinant

Let $A = \begin{pmatrix} 2 & 0 & 5 \\ 0 & 1 & 3 \\ -1 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 8 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{pmatrix}$

1. Find $det(A)$ by expanding along row 1 and along column 2.

2. Find $det(B)$

3. Let C be obtained from A by (i) exchanging rows 1 and 3 followed by (ii) adding 2*row 1 to row 3. Find $det(A)$.

4. Find $D = BA$ and find $det(D)$ two ways.