1. $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$

$\underset{A}{\text{row space}} = \text{span}\left(\{(1,2,-1),(2,0,1),(0,-1,4)\}\right) = W_1$ ← a subspace!

$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A'$

$\dim(W_1) = 3$ ? yes.

$\underset{A}{\text{row space}} = \underset{A'}{\text{row space}}$

So $W_1 = \text{span}\left(\{(1,0,0),(0,1,0),(0,0,1)\}\right)$ clearly a basis. So $\dim(W_1) = 3$

$A^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ -1 & 1 & 4 \end{pmatrix}$; $\text{rref}(A^T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\underset{A}{\text{col. space}} = \underset{A^T}{\text{row space}} = W_3 = \text{span}\left(\{(1,0,0),(0,1,0),(0,0,1)\}\right)$

$\dim(\text{col. space } A) = 3.$

$\boxed{\text{rank}(A) = 3}$

2. $B = \begin{pmatrix} 1 & 2 & -1 & 8 \\ 2 & 4 & 1 & 4 \\ 0 & -0 & 5 & 1 \\ 3 & 6 & 0 & 12 \\ -6 & -12 & 0 & -24 \end{pmatrix}$, $\underset{B}{\text{row space}} = \text{span}\left(\{(1,2,-1,8),(2,4,1,4),(0,0,5,1),(3,6,0,12),(-6,-12,0,-24)\}\right) = W_2$ ← also a subspace

$\dim(W_2) = \cancel{5}$ ? $= 3$

$B' = \text{rref}(B) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; $\underset{B}{\text{row space}} = \underset{B'}{\text{row space}}$; So $W_2 = \text{span}\left(\{(1,2,0,0),(0,0,1,0),(0,0,0,1)\}\right)$ obviously a basis. So

$B^T = \begin{pmatrix} 1 & 2 & 0 & 3 & -6 \\ 2 & 4 & 0 & 6 & -12 \\ -1 & 1 & 5 & 0 & 0 \\ 8 & 4 & 1 & 12 & -24 \end{pmatrix}$; $\underset{B}{\text{Col space}} = \underset{B^T}{\text{row space}}$; $\underset{B}{\text{Col space}} = \text{span}\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}\right) = W_3$, $\dim(W_3) = 3$

$\text{rref}(B^T) = \begin{pmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Note: $\underset{A}{\text{row space}} \subseteq \mathbb{R}^4$; $\underset{A}{\text{col space}} \subseteq \mathbb{R}^5$

$\text{rank}(B) = 3$

3. $D = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right) = \left(\begin{array}{c|c} A & \begin{matrix}0\\0\\0\end{matrix} \end{array}\right)$

#1 on front

(Think of A as coeff. matrix of homogeneous system.)

Int. 1

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 0 \\ 2c_1 + c_3 &= 0 \\ -c_2 + 4c_3 &= 0 \end{aligned}$$

row ops do NOT change solutions HERE,

so . . . . > . - > - - >

or

Int 2

$$c_1\begin{pmatrix}1\\2\\0\end{pmatrix} + c_2\begin{pmatrix}2\\0\\-1\end{pmatrix} + c_3\begin{pmatrix}-1\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Columns of A.

Row operations can't change solutions here either.

Again: $\mathrm{rref}(D) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ means $\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$ or $c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

or

$$\mathrm{rref}\left(\left[B \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right]\right) = \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & c_4 & \end{matrix} \\ \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \text{ means } \begin{aligned} c_1 + 2c_2 &= 0 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}$$

or $c_1\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + c_2\begin{pmatrix}2\\0\\0\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + c_4\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$

These are multiples. Go look ...

or $\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} = \begin{pmatrix}-2c_2\\c_2\\c_3\\c_4\end{pmatrix} = c_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + c_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$