1. Below is a homogeneous system of linear equations, the coefficient matrix A and the reduced echelon form of matrix A, called B. Answer the questions below.

$$\begin{cases} v + 2w + x + 2y + z = 0 \\ -v - 2w + x + \ y + z = 0 \\ 2v + 4w \quad\quad + \ y \quad\ \ = 0 \\ \quad\quad\quad\quad x + \ y + z = 0 \end{cases} \qquad A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ -1 & -2 & 1 & 1 & 1 \\ 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{array}{c} \begin{array}{ccccc} v & w & x & y & z \end{array} \\ \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(a) What is the rank of A? 3

(b) Find the set of solutions to the system and express the set in vector form.

$v + 2w = 0$
$x + z = 0$
$y = 0$

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2w \\ w \\ -z \\ 0 \\ z \end{pmatrix}; \quad \text{sol. set} = \left\{ w\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} : w, z \in \mathbb{R} \right\}$$

(c) Is the set of solutions a vector space? Why or why not?

Yes. $\text{Sol. set} = \text{span}\left(\left\{ \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{a}_1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{a}_2} \right\} \right)$ and span(S) is always a vector space.

(d) What is the *dimension* of the solution set? 2

Vectors $\vec{a}_1$ and $\vec{a}_2$ span the soln. set by definition. They are linearly independent b/c they have non zero entries in different positions.

2. (Theorem 3.13) Let A be an $m \times n$ matrix with rank r. What sort of number can r be? If A is the coefficient matrix of a homogeneous system, how many equations and how many unknowns are there? What can you say about the solution set?

- $r \leq n$ and $r \leq m$
- $[A \vdots \vec{0}]$ corresponds to a system of m equations and n unknowns.
- The solution set of the system with matrix form $[A \vdots \vec{0}]$ must have $n-r$ free variables and there for be a subspace of dimension $n-r$.

3. (Corollary 3.14) Let A be an $n \times n$ matrix. The followins statements are equivalent:

   (a) A has rank n
   (b) (what can you say about the rows?) The n rows are linearly independent.
   (c) (what can you say about the columns?) The n columns are linearly independent.
   (d) (what can you say about SoLE's with A as a coefficient matrix?) There is always exactly 1 solution.
   (e) (is A singular or nonsingular?) nonsingular.

Section 3.1.1

1. Let $V = \mathbb{R}^3$ and $W = \mathcal{P}_2$. Give an intuitive argument that these are not really different vector spaces.

The place holders $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ (or 1st, 2nd, 3rd coord) feel exactly the same as place holders $a+bx+cx^2$ (called constant, linear, and quadratic coefficients)

2. **Definition**: V, W vector spaces. $f: V \to W$ is an isomorphism if ① f is 1-1 and onto, ② $f(\vec{v}_1+\vec{v}_2) = f(\vec{v}_1)+f(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$ and ③ for all $r \in \mathbb{R}, \vec{v}_1 \in V$ $f(r\vec{v}_1) = rf(\vec{v}_1)$.

We say V and W are isomorphic vector spaces.

$\boxed{\text{Isomorphic} \equiv \text{effectively the same}}$

3. (Lemma 1.11)

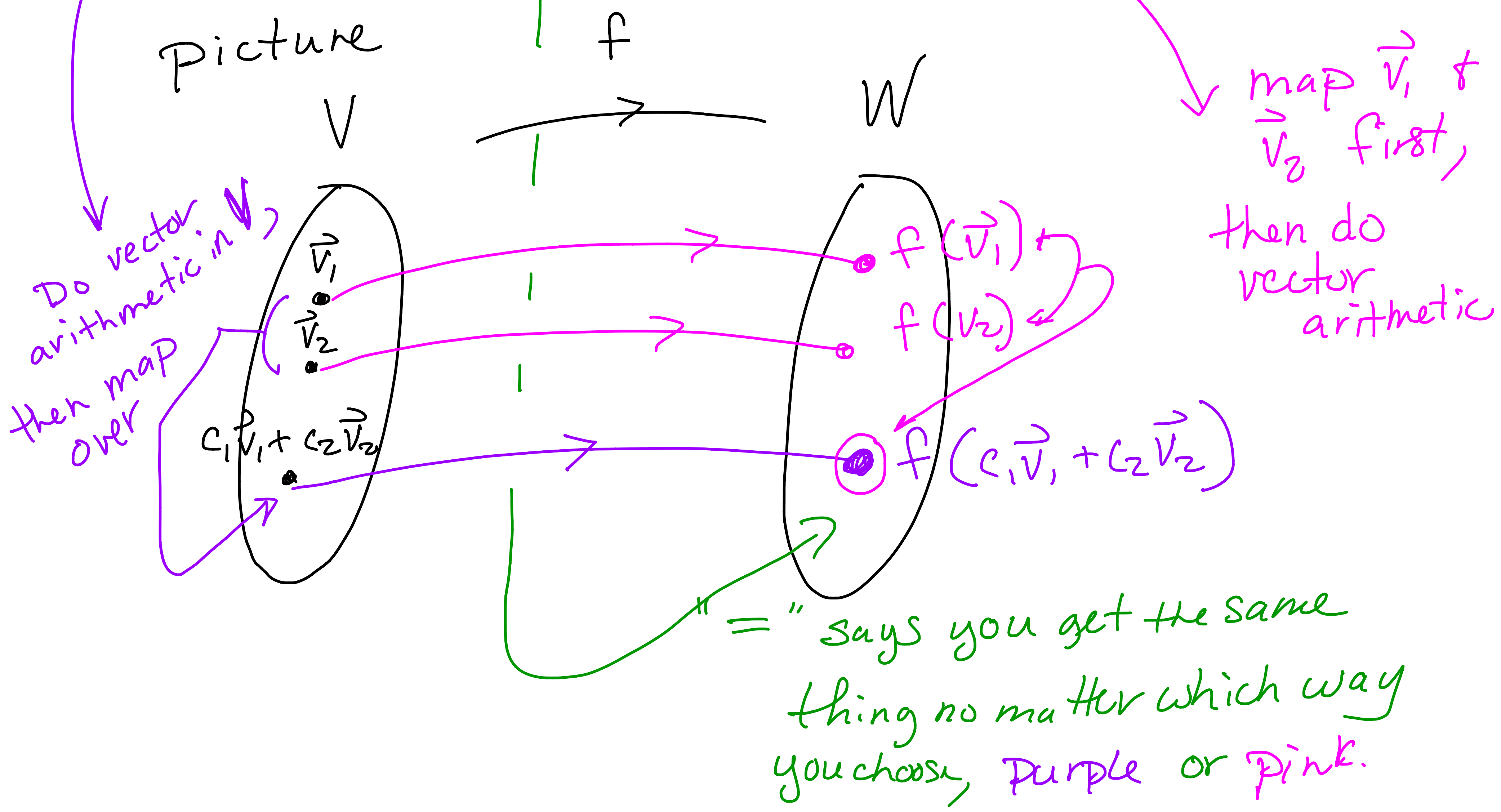
Instead of showing $f(\vec{v}_1+\vec{v}_2) = f(\vec{v}_1)+f(v_2)$ AND $f(r\vec{v}_1) = rf(\vec{v}_1)$

we can instead show

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) \quad \text{for all } c_1, c_2 \in \mathbb{R}, \ \vec{v}_1, \vec{v}_2 \in V.$$

Picture Version

$$\boxed{f(c_1\vec{v}_1+c_2\vec{v}_2)} \boxed{=} \boxed{c_1 f(\vec{v}_1)+c_2 f(\vec{v}_2)}$$ for all $c_1, c_2 \in \mathbb{R}$, $\vec{v}_1, \vec{v}_2 \in V$.



<u>Example</u>: Show $V=\mathbb{R}^3$ and $W=\mathbb{P}_2$ are isomorphic.

Pick a correspondence between V and W: $f\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=a+bx+cx^2$.

<u>Show f is 1-1</u>: Sppse $f\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=f\left(\begin{bmatrix}a'\\b'\\c'\end{bmatrix}\right)$. Then, $a+bx+cx^2=a'+b'x+c'x^2$.

So $a=a'$, $b=b'$, $c=c'$. So $\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}a'\\b'\\c'\end{bmatrix}$. So f is 1-1

<u>Show f is onto</u>: Let $a+bx+cx^2$ be any polynomial in $\mathbb{P}_2$.

Pick $\vec{v}=\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3$. Now $f\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=a+bx+cx^2$. So f is onto.

<u>Show f respects vector operations</u>: Let $c_1, c_2\in\mathbb{R}$ and $\begin{bmatrix}a\\b\\c\end{bmatrix}, \begin{bmatrix}a'\\b'\\c'\end{bmatrix}\in V=\mathbb{R}^3$.

$$f\left(c_1\begin{bmatrix}a\\b\\c\end{bmatrix}+c_2\begin{bmatrix}a'\\b'\\c'\end{bmatrix}\right)=f\left(\begin{bmatrix}c_1a+c_2a'\\c_1b+c_2b'\\c_1c+c_2c'\end{bmatrix}\right)=(c_1a+c_2a')+(c_1b+c_2b')x+(c_1c+c_2c')x^2$$

$$c_1f\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)+c_2f\left(\begin{bmatrix}a'\\b'\\c'\end{bmatrix}\right)=c_1(a+bx+cx^2)+c_2(a'+b'x+c'x^2)$$

$$=(c_1a+c_2a')+(c_1b+c_2b')x+(c_1c+c_2c')x^2$$

These are equal.