1. Show that $\mathbb{R}^3$ and $\mathcal{P}_2$ are isomorphic.

Example: Show $V=\mathbb{R}^3$ and $W=\mathcal{P}_2$ are isomorphic.

Pick a correspondence between V and W: $f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a+bx+cx^2$.

Show f is 1-1: Sppse $f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = f\left(\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}\right)$. Then, $a+bx+cx^2 = a'+b'x+c'x^2$.

So $a=a'$, $b=b'$, $c=c'$. So $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$. So f is 1-1

Show f is onto: Let $a+bx+cx^2$ be any polynomial in $\mathcal{P}_2$.

Pick $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Now $f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a+bx+cx^2$. So f is onto.

Show f respects vector operations: Let $c_1, c_2 \in \mathbb{R}$ and $\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} \in V = \mathbb{R}^3$.

$$f\left(c_1\begin{bmatrix} a \\ b \\ c \end{bmatrix} + c_2\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}\right) = f\left(\begin{bmatrix} c_1a+c_2a' \\ c_1b+c_2b' \\ c_1c+c_2c' \end{bmatrix}\right) = (c_1a+c_2a') + (c_1b+c_2b')x + (c_1c+c_2c')x^2$$

These are equal

$$c_1 f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) + c_2 f\left(\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}\right) = c_1(a+bx+cx^2) + c_2(a'+b'x+c'x^2)$$

$$= (c_1a+c_2a') + (c_1b+c_2b')x + (c_1c+c_2c')x^2$$

2. (Theorem 2.3) Vector spaces are isomorphic if and only if they have the same dimension.

Why?

Think about it as:

(i) If $V + W$ are isomorphic, then they have the same dimension.

AND

(ii) If $V + W$ have the same dimension, then they must be isomorphic.

(i) $V + W$ isomorphic $\Rightarrow f: V \to W$ isomorphism.

$\dim V = n \Rightarrow B = \langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \rangle$ is a basis of V.

We want $C = \langle \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \rangle$, a basis of W. How to find C?

We hope $C = \langle f(\vec{v}_1), f(\vec{v}_2), \ldots, f(\vec{v}_n) \rangle$. Will this really work?

What do we need to check?

Let $S = \{f(\vec{v}_1), f(\vec{v}_2), \ldots, f(\vec{v}_n)\}$, (a) Is S linearly independent? (b) Does S span W?

(a) $\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}}$ (1) if and only if $\boxed{c_1 f(\vec{v}_1) + c_2 f(v_2) + \ldots + c_n f(v_n) = 0}$ (2)

b/c f respects vector addition + scalar multiplication.

So if (1) has a unique (ie trivial) solution, so does (2). So S is linearly independent

(b) Pick $\vec{w} \in W$. Need to write $\vec{w}$ in terms of S. But f is onto.

So find $\vec{v} \in V$ so that $f(\vec{v}) = \vec{w}$. Since B is a basis of V,

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$. So $\vec{w} = f(\vec{v}) = c_1 f(v_1) + \ldots + c_n f(v_n)$.

So S spans.

Assume $\dim V = \dim W = n$.

How do we show V is isomorphic to W? find $f: V \to W$.

How do we find f?

If $B = \langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \rangle$ is a basis for V and

$C = \langle \vec{w}_1, \vec{w}_2, \dots, \vec{w}_n \rangle$ is a basis for W,

then define $f: V \to W$ by $f(\vec{v}_1) = \vec{w}_1$.

Wait... what about an arbitrary $v \in V$? $v = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

$$f(\vec{v}) = f(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) + \dots + c_n f(\vec{v}_n)$$
$$= c_1\vec{w}_1 + c_2\vec{w}_2 + \dots + c_n\vec{w}_n$$

If f an isomorphism? $\underbrace{\text{1-1, onto, } f(r_1 u_1 + r_2 u_2) = r_1 f(u_1) + r_2 f(u_2)}$,

These all hold b/c every vector in V or W can be written as a linear combination of B or C.