1. **Review** For vector spaces V and W, the function $f : V \to W$ is called an *isomorphism* if

f is 1-1, onto, and for every $r_1, r_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$

$f(r_1\vec{v}_1 + r_2\vec{v}_2) = r_1 f(\vec{v}_1) + r_2 f(\vec{v}_2)$

2. **Review** For vector spaces V and W, the function $f : V \to W$ is called a *homomorphism* or *linear map* if for every $r_1, r_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$, $f(r_1\vec{v}_1 + r_2\vec{v}_2) = r_1 f(\vec{v}_1) + r_2 f(\vec{v}_2)$

3. **Examples:**

(a) $f : \mathcal{P}_2 \to \mathbb{R}^2$ defined as $f(ax^2 + bx + c) = \begin{pmatrix} a \\ b + c \end{pmatrix}$ (or $ax^2 + bx + c \mapsto \begin{pmatrix} a \\ b + c \end{pmatrix}$)

Pick $a_1x^2 + b_1x + c_1,\ a_2x^2 + b_2x + c_2 \in \mathcal{P}_2$ and $r_1, r_2 \in \mathbb{R}$.

$LHS = f(r_1(a_1x^2 + b_1x + c_1) + r_2(a_2x^2 + b_2x + c_2)) = f((r_1a_1 + r_2a_2)x^2 + (r_1b_1 + r_2b_2)x + r_1c_1 + r_2c_2)$

$= \begin{pmatrix} ra_1 + r_2a_2 \\ r_1b_1 + r_2b_2 + r_1c_1 + r_2c_2 \end{pmatrix}$ These are equal.

$RHS = r_1 f(a_1x^2 + b_1x + c_1) + r_2 f(a_2x^2 + b_2x + c_2) = r_1\begin{pmatrix} a_1 \\ b_1 + c_1 \end{pmatrix} + r_2\begin{pmatrix} a_2 \\ b_2 + c_2 \end{pmatrix} = \begin{pmatrix} r_1a_1 + r_2a_2 \\ r_1(b_1 + c_1) + r_2(b_2 + c_2) \end{pmatrix}$

(b) The *projection* $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$

Pick $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in \mathbb{R}^3$, $r_1, r_2 \in \mathbb{R}$.

$\pi\left(r_1\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} r_1x_1 + r_2x_2 \\ r_1y_1 + r_2y_2 \end{pmatrix}$; $r_1\pi\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2\pi\begin{pmatrix} x_2 \\ y_2 \\ z_1 \end{pmatrix} = \begin{pmatrix} r_1x_1 + r_2x_2 \\ r_1y_1 + r_2y_2 \end{pmatrix}$

These are equal.

Domain Codomain

(c) The zero homomorphism $z : \mathbb{R}^4 \to \mathcal{M}_{2\times 2}$

$z\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. ($z$ maps every vector to the zero vector of the codomain.)

$z(r_1\vec{v}_1 + r_2\vec{v}_2) = \vec{0}$; $r_1 z(\vec{r}_1) + r_2 z(\vec{v}_2) = r_1\vec{0} + r_2\vec{0} = \vec{0}$.

4. Lemma 1.6:

If $f: V \to W$ is a homomorphism, then $f(\vec{0}_V) = \vec{0}_W$.

~~5. Lemma 1.7:~~

6. Theorem 1.9: A homomorphism $f: V \to W$ can be defined by the image of the basis B of V.

7. Consequence of Theorem 1.9: Define a homomorphism h from $\mathcal{P}_2$ to $\mathcal{M}_{2\times 2}$ by defining h on a basis of $\mathcal{P}_2$ and demonstrating how h operates on an arbitrary element of $\mathcal{P}_2$.

Pick $B = \langle 1, x, x^2 \rangle$. Define $h(1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $h(x) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $h(x^2) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

(Pick anything ...

Now $h(2-3x+4x^2) = h(2(1)-3(x)+4(x^2)) = 2h(1) - 3h(x) + 4(h(x^2))$

$= 2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & -7 \end{bmatrix}$

8. Definition 1.12

A homomorphism $f: \underbrace{V \to V}_{\text{same vector space.}}$ is called a <u>linear transformation</u>.

Ex $f: \mathbb{R}^3 \to \mathbb{R}^3$ by $(x,y,z) \mapsto (2x+y, z, z)$

Are any of these 1-1, onto?