1. **Review** For vector spaces V and W, the function $f : V \to W$ is called a *homomorphism* or *linear map* if

$$f(r_1\vec{v}_1 + r_2\vec{v}_2) = r_1 f(\vec{v}_1) + r_2 f(\vec{v}_2) \text{ for all } r_1, r_2 \in \mathbb{R}, \vec{v}_1, \vec{v}_2 \in V.$$

2. (Some preliminary terminology) Assume $f : V \to W$, $S \subseteq V$, and $T \subseteq W$.

   image: for $v \in V$, the image of v is $f(v)$, the image of S, denoted $f(S) = \{f(s) : s \in S\}$

   So range of f = $\mathcal{R}(f) = f(V)$

   inverse image: (preimage)

   for $w \in W$, the inverse image of w is $f^{-1}(w) = \{v \in V : h(v) = w\}$

   inverse function:

   f^{-1} is a function so that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

   - For any function f, we can find $f^{-1}(s)$ but $f^{-1}(x)$ may not be a function. (We need f to be 1-1 for f^{-1} to be a function.

3. (Lemma 2.1, Definition 2.2, and some notation) Assume $f : V \to W$ is a linear map between vector spaces V and W and assume V' is a subspace of V.

   - $f(V')$ is a subspace of W.
   - $f(V)$ is a subspace of W.
   - range space of f is $\mathcal{R}(f)$ w/ inherited vector operation from W.
   - dimension of $(\mathcal{R}(f))$ = dimension$(f(V))$ = dim. of range space

     = the rank of the map f

4. (Lemma 2.10, Definition 2.11, and some notation) Assume $f : V \to W$ is a linear map between vector spaces V and W and assume W' is a subspace of W.

- $f^{-1}(W')$ is a subspace of V
- $f^{-1}(\{\vec{0}_W\})$ is a subspace of V called the <u>null space</u> (or <u>kernel</u>) of f.
  Denoted $\mathcal{N}(f)$.
- $\dim(\mathcal{N}(f))$ is the <u>nullity of f.</u>

5. (Theorem 2.14 and Corollary 2.17) Assume $f : V \to W$ is a linear map between vector spaces V and W.

- $\dim V = \operatorname{rank}(f) + \operatorname{nullity}(f)$
- $\operatorname{rank} f \leq \dim V$
- $\operatorname{rank} f = \dim V \iff \operatorname{nullity}(f) = 0$
- $\operatorname{nullity}(f) = 0 \iff f : V \to \mathcal{R}(V)$ is an isomorphism

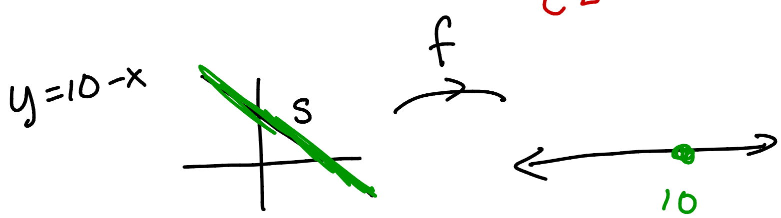
6. (Theorem 2.20) Assume $f : V \to W$ is a linear map between vector spaces V and W and $dim(V) = n$. The following are equivalent statements.

1. f is 1-1
2. $f^{-1} : f(V) \to V$ is a function
3. $\operatorname{rank}(f) = n$
4. $\operatorname{nullity}(f) = 0$ (i.e. $\mathcal{N}(f) = \{\vec{0}_V\}$
5. If $\langle \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \rangle$ is a basis of V, then $\langle f(\vec{b}_1), f(\vec{b}_2), \ldots, f(\vec{b}_n) \rangle$ is a basis of $\mathcal{R}(f)$

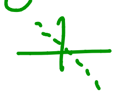
Examples:

1. $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y$

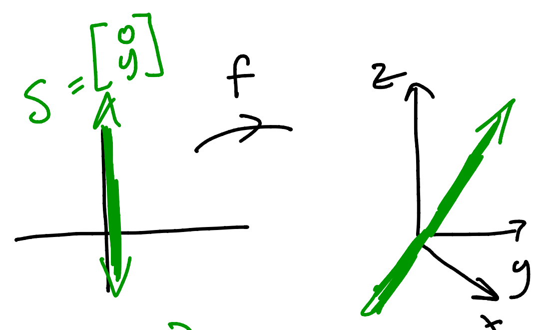
- Find the image of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 1+2=3$
- Find the image of $S = \left\{ \begin{bmatrix} x \\ 10-x \end{bmatrix} : x \in \mathbb{R} \right\}$. $f\left(\begin{bmatrix} x \\ 10-x \end{bmatrix}\right) = 10$



- Find the inverse image of 10. $S = \left\{ \begin{bmatrix} x \\ 10-x \end{bmatrix} : x \in \mathbb{R} \right\}$
- Find the inverse image of 0. $S = \left\{ \begin{bmatrix} x \\ -x \end{bmatrix} : x \in \mathbb{R} \right\} = \left\{ x\begin{bmatrix} 1 \\ -1 \end{bmatrix} : x \in \mathbb{R} \right\}$



- $\mathcal{R}(f) = \mathbb{R}$. Dimension $=1$
- $\mathcal{N}(f) = \left\{ r\begin{bmatrix} 1 \\ -1 \end{bmatrix} : r \in \mathbb{R} \right\}$, nullity $=1$



2. $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ y \end{bmatrix}$

- Image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$
- image of $S = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$. $f(S) = \left\{ \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$
- Find the inverse image of $T = \left\{ \begin{bmatrix} 0 \\ a \\ a \end{bmatrix} : a \in \mathbb{R} \right\}$.
- Find the inverse image of $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
- $\mathcal{R}(f) = \left\{ \begin{bmatrix} x \\ y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\} = \left\{ x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}$

  dimension $=2$. ($\mathcal{R}(f)$ is a plane determined by points $(0,0,0)$, $(1,0,0)$ and $(0,1,1)$)
- $\mathcal{N}(f) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, nullity $=0$.