1. The Big Idea: A linear map between vector spaces can always be described as a matrix which can be used to find the image of vectors using the matrix-vector product. (a thinking-free automation)

2. The Big Idea in a formal way:
   Let $f : V \to W$ be a **linear map** between **vector spaces** $V$ and $W$ with **bases** $B = \langle \vec{b_1}, \vec{b_2}, \cdots, \vec{b_n} \rangle$, (for $V$ of dimension $n$) and $D = \langle \vec{d_1}, \vec{d_2}, \cdots, \vec{d_m} \rangle$, (for $W$ of dimension $m$).

   Then the matrix $M$ representing the linear map has dimensions $\underbrace{m}_{\text{codomain}} \times \underbrace{n}_{\text{domain}}$

   with columns formed as follows

$$M = \begin{bmatrix} | & | & & | \\ \operatorname{rep}_{\underline{\underline{D}}}(f(\vec{b_1})) & \operatorname{rep}_{\underline{\underline{D}}}(f(\vec{b_2})) & \cdots & \operatorname{rep}_{\underline{\underline{D}}}(f(\vec{b_n})) \\ | & | & & | \end{bmatrix} = \operatorname{rep}_{B,D}(f)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{column } 1 & \text{column } 2 & \text{column } n \\ \text{w/ } m \text{ enteries} & \text{w/ } m \text{ enteries} & \text{w/ } m \text{ enteries} \end{array}$$

   The image of a vector $\vec{v} \in V$ can be found by

$$\operatorname{rep}_D(f(\vec{v})) = M \cdot \operatorname{rep}_B(\vec{v}) = \operatorname{rep}_{B,D}(f) \cdot \operatorname{rep}_B(\vec{v})$$

$$\begin{array}{r} \text{row1} \to \\ \text{row2} \to \\ \\ \text{row } m \to \end{array} \begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,n} \\ h_{2,1} & h_{2,2} & & h_{2,n} \\ \vdots & \vdots & & \vdots \\ h_{m,n} & h_{m,2} & & h_{m,n} \end{bmatrix} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix}_B}_{\boxed{n} \text{ enteries}} = \begin{pmatrix} h_{1,1}c_1 + h_{1,2}c_2 + \ldots + h_{1,n}c_n \\ h_{2,1}c_1 + h_{2,2}c_2 + \ldots + h_{2,n}c_n \\ \vdots \\ h_{m,1} \cdot c_1 + h_{m,2} \cdot c_2 + \ldots + h_{m,n} \cdot c_n \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{col } 1 & \text{col2} & \text{coln} \end{array}$$

3. Fact we will use: Any linear map $f : V \to W$ between vector spaces can be determined by

   the images of a basis of $V$.

4. Example: $f : \mathbb{R}^3 \to \mathbb{R}^2$

bases: $B = \left\langle \vec{b_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \vec{b_2} = \begin{pmatrix}0\\2\\0\end{pmatrix}, \vec{b_3} = \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle$ and $D = \left\langle \vec{d_1} = \begin{pmatrix}1\\0\end{pmatrix}, \vec{d_2} = \begin{pmatrix}-1\\-1\end{pmatrix} \right\rangle$

image of basis vectors: $f(\vec{b_1}) = \begin{pmatrix}1\\-1\end{pmatrix}$, $f(\vec{b_2}) = \begin{pmatrix}0\\2\end{pmatrix}$, $f(\vec{b_3}) = \begin{pmatrix}0\\0\end{pmatrix}$

All of this is given to us...

$\vec{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

(a) Find the image of $\vec{v}$ under $f$ and express its image with respect to the basis $D$ **via actual thinking...**

① Find $\text{rep}_B(\vec{v}) = \begin{pmatrix}1\\-1\\3\end{pmatrix}_B$.

Find $c_1, c_2, c_3$ so that $\begin{pmatrix}1\\2\\3\end{pmatrix} = c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\2\\0\end{pmatrix} + c_3\begin{pmatrix}0\\1\\1\end{pmatrix}$

Solve $\left[\begin{array}{ccc|c}1&0&0&1\\1&2&1&2\\0&0&1&3\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&3\end{array}\right]$

(check correctness!)

$1\begin{pmatrix}1\\1\\0\end{pmatrix} - 1\begin{pmatrix}0\\2\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

② Find $f\left(\text{rep}_B(\vec{v})\right) = \begin{pmatrix}1\\-3\end{pmatrix}$

$= f(1\cdot\vec{b_1} + (-1)\vec{b_2} + 3\vec{b_3}) = 1\cdot f(\vec{b_1}) - 1\cdot f(\vec{b_2}) + 3f(\vec{b_3}) = 1\begin{pmatrix}1\\-1\end{pmatrix} - 1\begin{pmatrix}0\\2\end{pmatrix} + 3\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$

③ Find $\text{rep}_D\left(f(v)\right) = \text{rep}_D\left(\begin{pmatrix}1\\-3\end{pmatrix}\right) = \begin{pmatrix}4\\3\end{pmatrix}$

Find $c_1, c_2$ so that $c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$ or Solve $\left[\begin{array}{cc|c}1&-1&1\\0&-1&-3\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c}1&0&4\\0&1&3\end{array}\right]$

④ ANSWER: $\text{rep}_D\left(f(\vec{v})\right) = \begin{pmatrix}4\\3\end{pmatrix}_D$.

(b) Find the image of $\vec{v}$ under $f$ and express its image with respect to the basis $D$ **via automation.**

① Find $\text{rep}_D\left(f(\vec{b_i})\right)$ for all basis vectors:

quick check $-2\begin{pmatrix}1\\0\end{pmatrix} - 2\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$ ✓

$\text{rep}_D\left(f(\vec{b_1})\right) = \text{rep}_D\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\1\end{pmatrix}_D$; $\text{rep}_D\left(f(\vec{b_2})\right) = \text{rep}_D\left(\begin{pmatrix}0\\2\end{pmatrix}\right) = \begin{pmatrix}-2\\-2\end{pmatrix}_D$

$\text{rep}_D\left(f(\vec{b_3})\right) = \text{rep}_D\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}0\\0\end{pmatrix}_D$

② $M = \begin{bmatrix}2&-2&0\\1&-2&0\end{bmatrix}$

③ $\text{rep}_D\left(f(\vec{v})\right) = M \cdot \text{rep}_B(\vec{v}) = \begin{bmatrix}2&-2&0\\1&-2&0\end{bmatrix} \cdot \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{bmatrix}2\cdot1+(-2)\cdot(-1)+0\cdot3\\1\cdot1+(-2)(-1)+0\cdot3\end{bmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$

dotproduct of rows w/ $\vec{v}$

These are the same!