1. The Big Idea from 3.3.1: A linear map between vector spaces can always be described as a matrix which can be used to find the image of vectors using the matrix-vector product. (a thinking-free automation)

2. Review Example: $h : \mathbb{R}^3 \to \mathbb{R}^2$ by $h\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$, $h\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$, $h\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$. Assume the basis for $\mathbb{R}^3$ is $B = \langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \rangle$ and $\mathcal{E}_2$ for $\mathbb{R}^2$. Find $\text{rep}_{B,\mathcal{E}_2}(h)$ and use it to find the image of $\vec{v} = [1, 2, 3]$.

$\text{rep}_{B,E_2}(h) = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\end{bmatrix}$; $\text{rep}_B\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}-2\\-1\\3\end{bmatrix}$.

So $h(\vec{v})_{\mathcal{E}_2} = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}-2\\-1\\3\end{bmatrix} = \begin{bmatrix}-2\cdot 0 + (-1)(-1) + (0)(3)\\(1)(-2) + (0)(-1) + (0)(3)\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$

exactly what we thought!

3. The Big Idea from 3.3.2: Given any $m \times n$ matrix M, we can view M as a linear map between two vector spaces $V \to W$ of dimensions n and m respectively with respect to any pair of bases.

4. Simple Example: Let $M = \begin{bmatrix}1 & 3\\0 & -1\\1 & 1\\2 & 3\end{bmatrix}$. M represents map from V (dim 2) to W (dim 4)

V has basis $B = \langle 1, x \rangle$, W has basis $D = \langle 1, x, x^2, x^2+x^3 \rangle$

$\vec{v} \in V$ s.t. $(\vec{v})_B = \begin{pmatrix}2\\3\end{pmatrix}$ (or $\vec{v} = 2+3x$).

Then $h(\vec{v}) = \begin{bmatrix}1 & 3\\0 & -1\\1 & 1\\2 & 3\end{bmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \begin{bmatrix}2+9\\-3\\2+3\\4+9\end{bmatrix} = \begin{bmatrix}11\\-3\\5\\13\end{bmatrix}_D = 11 - 3x + 5x^2 + 13(x^2+x^3)$

Obviously, different bases give different maps!

5. Let M be an $m \times n$ matrix representing the linear map $h : V \to W$, for vector spaces V and W of dimensions n and m respectively. (There is an underlying assumption that bases for V and W are known.)

   (a) **Theorem 2.4:** Rank of M = rank of h

Why? Recall that rank of h = dimension of the range space.
Let $B = \langle \vec{b}_1, \vec{b}_2, \dots, \vec{b}_n \rangle$, basis for V. Then $S = \{h(\vec{b}_1), h(\vec{b}_2), \dots, h(\vec{b}_n)\}$ must SPAN $\mathcal{R}(h)$
(Why isn't it a basis?) But we can find a basis for $\mathcal{R}(h)$ by, one-by-one, deleting from S a vector linearly dependent on other vectors in S.
But now we observe that the columns of M are the elements of S. So span(S) = $\mathcal{R}(h)$ = column space of M.
And rank M = rank col. space of M.

   (b) **Corollary 2.6**

   - h is onto if and only if rank of M is $m = \dim(W)$.
   - h is one-to-one if and only if rank of M is $n = \dim(V)$

Revisit #4.

$$M = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

rank $M = 2$. So rank $h = 2 = \dim(\mathcal{R}(h))$
So nullity $h = 0$
So h is 1-1 and not onto.

   (c) **Lemma 2.9:** h is an isomorphism if and only of M is nonsingular.

$h: V \to W$ isom $\Leftrightarrow$ h is 1-1 and onto $\Leftrightarrow$ M is square $n \times n$
rank = $\dim V = \dim W = n$

Hence: h is nonsingular. ← terminology
If the matrix M associated w/ homomorphism h is a square singular matrix, then h is singular (+ h is not 1-1 and not onto!)

6. Give examples of singular and nonsingular homomorphisms from $\mathbb{R}^3 \to \mathbb{R}^3$.

Singular

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 2

nonsingular

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

rank = 3