# Section 3.4.2 and 3.4.3: Composition of Linear Maps and Matrix Multiplication (Day 2)

1. **Take-aways from Monday** Let $f : V \to W$ and $g : W \to Y$ be linear maps with matrix representations $A$ and $B$ respectively. Then,

   - the matrix representation of $(g \circ f) :$ $V \to Y$ is $BA$ with dimension $\dim(Y) \times \dim(V)$
   - the function $(g \circ f)$ is a linear map. (ie the composition of linear maps is also linear!)
   - Function composition - matrix multiplication is NOT commutative.
   - Function composition - matrix multiplication IS associative.

     $$(f \circ g \circ h) = f \circ (g \circ h) = (f \circ g) \circ h$$

     $$ABC = A(BC) = (AB)C$$

   - Function composition - matrix multiplication is distributive.

     $$f \circ (g+h) = (f \circ g) + (f \circ h)$$

     $$A(B+C) = AB + AC$$

     Note $f, g, h$ being linear maps really matters here!
     - $\sin(e^x + x^2) \neq \sin(e^x) + \sin(x^2)$
     - $5(8x + 7x) = 5(8x) + 5(7x)$

2. Terminology

   (a) main diagonal

   $A$ is square $n \times n$ matrix

   main diagonal : $a_{ii}$ entries

   $$\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \; a_{nn} \end{bmatrix}$$

   (b) identity matrix

   $I_n$ has 1's on main diagonal, 0's elsewhere

   $$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) diagonal matrix

has 0's off the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 4 \\ 3 & -9 \end{bmatrix}$$

If $D_n$ is diagonal and $A$ is $n \times m$, what does $D_n A$ look like? $A D_n$?

Row $i$ of $A$ is multiplied by $d_{ii}$

(d) permutation matrix

has exactly one 1 in every row + every column and zeros elsewhere.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad I_3 \qquad \underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

If $P$ is $n \times n$ permutation matrix, $A$ is $n \times m$ matrix, then $PA$ permutes the rows of $A$.

$AP$? permutes columns

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{bmatrix}$$

(e) elementary (reduction) matrices

matrices that perform elementary row operations.

Ex: Construct $M$ that adds $2 \times$ row 1 to row 2 + leaves everything else fixed

$$\underset{M}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 7 & 8 & 9 \end{bmatrix}$$