1. **Example** Let $f:\mathbb{R}^3\to\mathbb{R}^2$ and $g:\mathbb{R}^2\to\mathbb{R}^4$ be linear maps with matrix representations

$$A=\begin{bmatrix}1&-1&0\\0&2&1\end{bmatrix}\text{ and }B=\begin{bmatrix}2&1\\0&0\\0&1\\0&2\end{bmatrix}\text{ (with standard bases }\mathcal{E}_3,\mathcal{E}_2,\text{ and }\mathcal{E}_4.)\text{ Let }\vec{v}=\begin{pmatrix}2\\1\\-1\end{pmatrix}.$$

Find $(g\circ f)(\vec{v})$.

gut checks

- Does $g\circ f$ make sense?
- Does $f\circ g$?
- Which matrix goes with which function?

way 1

$$f(\vec{v})=A\vec{v}=\begin{bmatrix}1&-1&0\\0&2&1\end{bmatrix}\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{bmatrix}2-1+0\\0+2-1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$g(f(\vec{v}))=g\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=B\cdot f(\vec{v})=\begin{bmatrix}2&1\\0&0\\0&1\\0&2\end{bmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{bmatrix}2+1\\0\\1\\2\end{bmatrix}=\begin{pmatrix}3\\0\\1\\2\end{pmatrix}$$

way 2

$$BA=\underset{4\times 2}{\begin{bmatrix}2&1\\0&0\\0&1\\0&2\end{bmatrix}}\underset{2\times 3}{\begin{bmatrix}1&-1&0\\0&2&1\end{bmatrix}}=\underset{4\times 3}{\begin{bmatrix}2+0&-2+2&0+1\\0&0&0\\0&2&1\\0&4&2\end{bmatrix}}=\begin{bmatrix}2&0&1\\0&0&0\\0&2&1\\0&4&2\end{bmatrix}$$

$$(BA)\vec{v}=\begin{bmatrix}2&0&1\\0&0&0\\0&2&1\\0&4&2\end{bmatrix}\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}4+0-1\\0\\2-1\\4-2\end{pmatrix}=\begin{pmatrix}3\\0\\1\\2\end{pmatrix}$$

The same!

$$A=[a_{ij}]=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}=\begin{array}{cc}&j\\i&\begin{bmatrix}&&\\&a_{ij}&\\&&\end{bmatrix}\end{array}$$

$$C=A\cdot B=\begin{array}{cc}&\\i&\begin{bmatrix}&&\\a_{i1}&a_{i2}\cdots&a_{in}\\&&\end{bmatrix}\\&A\end{array}\begin{array}{c}j\\\begin{bmatrix}&b_{1j}&\\&b_{2j}&\\&\vdots&\\&b_{nj}&\end{bmatrix}\\B\end{array}=\begin{array}{cc}&j\\i&\begin{bmatrix}&&\\&C_{ij}&\\&&\end{bmatrix}\end{array}$$

$$C_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\cdots+a_{in}b_{nj}$$

On prev. page

2. **Definition 2.3 of Matrix Multiplication** $A = [a_{ij}]$ is $m \times n$ and $B = [b_{ij}]$ is $n \times p$. Then $C = [c_{ij}] = AB$ is defined as

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}.$$

3. **Example:** Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 10 & 0 & 4 \\ 0 & 3 & -1 \end{bmatrix}$. Find the following products or state that they are undefined.

(a) AB

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 1\cdot 1 - 1\cdot 1 & (1)(-1)+(1)(3)+(-1)0 \\ 0+0+1 & 0+0+0 \\ 4+1+0 & -2+3+0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 5 & 1 \end{bmatrix}$$

(b) BA

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \Big[$$

not going to work...

(c) A^2

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+0-2 & 1+0-1 & -1+1+0 \\ 0+0+2 & 0+0+1 & 0+0+0 \\ 2+0+0 & 2+0+0 & -2+1+0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix}$$

(d) BC

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 10 & 0 & 4 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 20-0 & 0-3 & 4+1 \\ 10 & 9 & 4-3 \\ 10 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 20 & -3 & 5 \\ 10 & 9 & 1 \\ 10 & 0 & 4 \end{bmatrix}$$

(e) CB

$$\begin{bmatrix} 10 & 0 & 4 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 20+0-1 & -10+0+0 \\ 0+3-1 & 0+9+0 \end{bmatrix} = \begin{bmatrix} 19 & -10 \\ 2 & 9 \end{bmatrix}$$

4. Observations: • Matrix multiplication is not commutative.

• The matrix representing $(g \circ f)$ is the product BA where A represents f and B represents g.