SECTION 3.4.4 INVERSES

1. The function $f : V \to W$, a linear map with matrix representation A, has an inverse if and only if

   - f is 1-1 and onto
   - f is an isomorphism
   - A is nonsingular

2. If the function $f : V \to W$, a linear map with matrix representation A, has inverse $f^{-1} : W \to V$ with matrix representation B, then $AB = I_n$ and then $BA = I_n$ (where $n = \dim(V) = \dim W$)

   Call $B = A^{-1}$.

   (So $AA^{-1} = A^{-1}A = I_n$)

   So some basic algebra is possible!

   Ex| If A is nonsingular, then solve $AC = B$ for matrix C.

   $$A^{-1}AC = A^{-1}B$$
   $$I_n C = A^{-1}B$$
   $$\boxed{C = A^{-1}B}$$

3. If A is a nonsingular $n \times n$ matrix, then $\text{rref}(A) = I_n$.

   So $A \xrightarrow[\text{ops}]{\text{row}} I_n$ OR $\underbrace{R_k \cdots R_2 R_1}_{\text{multiply by elementary reduction matrices.}} A = I_n$

   But now:

   $$\underbrace{(R_k \cdots R_2 R_1)}_{\text{This \underline{is} } A^{-1}.} A = I_n$$

   So, if we could capture or keep track of row operations we can find A^{-1}.

   ↓ see next algorithm.

4. Find A^{-1} for $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & -1 & -2 \end{pmatrix}$.

augmented matrix:

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/4 & 1/4 & 3/4 \\ 0 & 1 & 0 & 3/4 & -5/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & 3/4 & 1/4 \end{array}\right]$$

$A \;\vdots\; I_3$

we knew this!

captured row ops

So $A^{-1} = \begin{bmatrix} 1/4 & 1/4 & 3/4 \\ 3/4 & -5/4 & -3/4 \\ -1/4 & 3/4 & 1/4 \end{bmatrix}$

5. Solve the system of equations $\begin{cases} x + 2y + 3z = 8 \\ \quad\;\; y + 3z = -4 \\ x - \;\; y - 2z = 0 \end{cases}$

Solt: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$ or $A\vec{x} = \vec{b}$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$

So $A^{-1}A\vec{x} = A^{-1}\vec{b}$ or $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1/4 & 1/4 & 3/4 \\ 3/4 & -5/4 & -3/4 \\ -1/4 & 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \\ -5 \end{bmatrix}$

$1 + 22 - 15 = 23 - 15 = 8$ ✓

$11 - 15 = -4$ ✓

$1 - 11 - 2(-5) = 11 - 11 = 0$ ✓

Most important, solution algorithm is easy for all choices of $\vec{b}$.