Section 3.5.1 and 3.5.2 Change of Basis
Do an example on your own

Example:

$$\mathcal{E}_3 = \left\langle \vec{e_1} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{e_2} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{e_3} = \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle \qquad B = \left\langle \vec{b_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \vec{b_2} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \vec{b_3} = \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle$$

$$h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ has matrix representation } A = \begin{pmatrix}5 & -3 & -6\\3 & -1 & -6\\3 & -3 & -4\end{pmatrix} \text{ w.r.t } \mathcal{E}_3$$

1. Find $S = \text{Rep}_{B,\mathcal{E}_3}(id)$, the matrix that translates vectors in base B to vectors in $\mathcal{E}_3$.

$$S = \begin{bmatrix} | & | & | \\ (\vec{b}_1)_{\mathcal{E}_3} & (\vec{b}_2)_{\mathcal{E}_3} & (\vec{b}_3)_{\mathcal{E}_3} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

2. Find $T = \text{Rep}_{\mathcal{E}_3,B}(id)$, the matrix that translates vectors in base $\mathcal{E}_3$ to vectors in base B.

<u>way 1</u> $T = S^{-1}$ or <u>way 2</u> $T = \begin{bmatrix} | & | & | \\ (\vec{e}_1)_B & (\vec{e}_2)_B & (\vec{e}_3)_B \\ | & | & | \end{bmatrix}$ In either case:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{c|ccc} & .5 & .5 & -1 \\ I_3 & -.5 & .5 & 1 \\ & .5 & -.5 & 0 \end{array}\right], \quad T = \begin{bmatrix} 1/2 & 1/2 & -1 \\ -1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

3. Find $C = \text{Rep}_{B,B}(h)$, the matrix representation of h with respect to basis B.

idea: $(\vec{v})_B \xrightarrow[S]{B \to \mathcal{E}_3} (\vec{v})_{\mathcal{E}_3} \xrightarrow[A]{h} (h(\vec{v}))_{\mathcal{E}_3} \xrightarrow[T]{\mathcal{E}_3 \to B} (h(\vec{v}))_B$ // So $TAS\vec{v}$

order of operations. $\longrightarrow$

$\longleftarrow$ order of operations

$$C = T \cdot A \cdot S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4. Check that your answers are correct by mapping the vector $\vec{v} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}_{\mathcal{E}_3}$ using matrix A and using matrix C and showing that those output vectors are the same.

① Using A: $h(\vec{v}) = A\vec{v} = \begin{pmatrix} 5 & -3 & -6 \\ 3 & -1 & -6 \\ 3 & -3 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} -20 \\ -28 \\ -12 \end{pmatrix}_{\mathcal{E}_2}$

② Using C: $\vec{v} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}_{\mathcal{E}_3} = \begin{pmatrix} -6 \\ 4 \\ 2 \end{pmatrix}_B$

$$\left(h(\vec{v})\right)_B = C \cdot \vec{v}_B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -16 \\ 4 \end{bmatrix}_B$$

③ Are they the same?

translate $\begin{pmatrix} -20 \\ -28 \\ -12 \end{pmatrix}_{\mathcal{E}_3} \rightarrow \begin{pmatrix} \\ \\ \\ \end{pmatrix}_B$ using T

$$(T)\begin{pmatrix} -20 \\ -28 \\ -12 \end{pmatrix}_{\mathcal{E}_3} = \begin{pmatrix} -12 \\ -16 \\ 4 \end{pmatrix}_B$$

$\boxed{\text{OR}}$

translate $\begin{pmatrix} -12 \\ -16 \\ 4 \end{pmatrix}_B \rightarrow \begin{pmatrix} \\ \\ \\ \end{pmatrix}_{\mathcal{E}_3}$ using S

$$(S)\begin{pmatrix} -12 \\ -16 \\ 4 \end{pmatrix}_B = \begin{pmatrix} -20 \\ -28 \\ -12 \end{pmatrix}_{\mathcal{E}_3} \quad \text{YaY!}$$

5. Draw a "potato picture" of what the matrices A, S, T, and C represent.