Goals:

- Know Terminology: linear combination, linear equation, coefficients, constant, a system of linear equations, a solution to a system of linear equations, elementary row operations
- Understand an algorithm: Gauss's Method (or Gaussian Elimination). Understanding an algorithm means knowing *when* to apply it, *how* to apply it and correctly *interpreting* the results.

1. linear combination

ex: $x_1 + \sqrt{5}x_2 - \frac{1}{\pi}x_3$

NOT linear

$x_1 + x_2^2 + x_3$ or $x_1 + 5x_1x_2 + x_2$

general: $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n$

2. linear equation

ex: $x_1 + \sqrt{5}x_2 - \frac{1}{\pi}x_3 = 8$

general: $a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$

← coefficient

← variable

← constant

3. system of linear equations

$\boxed{A}$
$$\begin{aligned} x + y &= 5 \\ x - y &= 8 \end{aligned}$$

or

$\boxed{B}$
$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ 5x_1 & & & - & 5x_3 & = & 10 \end{array}$$

4. a solutions to a system of linear equations

Claim

$\boxed{A}$ $x = \frac{13}{2}, y = \frac{-3}{2}$ is a solution. (Check!)

for $\boxed{B}$ $x=1, y=0, z=-1$ is a solution (Also, easily checked!)

5. elementary row operations

- multipy equ. (row) by a constant
- reorder rows/equations
- add a multiple of one row to another.

6. Gauss's Method

Goal: Find all solutions (if any exist!) to a system of equations.

Steps: Change one system of equations into a new system of equations using only elementary row operations such that the new system is easy to solve. In particular, try eliminating leading coefficients one by one.

Interpetation: The solution to the new system is the same as the solution of the original system. Solve the (easier) new system.

"ϱ" pronounced "rho" or "row"

$-5x_1 \quad +10x_2 \quad -5x_3$ $\qquad\qquad -10x_2 + 40x_3 = -40$

7. Example 1:
$$\begin{array}{lrcrcrcl} \varrho_1 & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \varrho_2 & & & 2x_2 & - & 8x_3 & = & 8 \\ \varrho_3 & 5x_1 & & & - & 5x_3 & = & 10 \end{array} \xrightarrow{\varrho_3 - 5\varrho_1 \mapsto \varrho_3} \begin{array}{rl} x_1 - 2x_2 + x_3 = 0 & \varrho_1 \\ 2x_2 - 8x_3 = 8 & \varrho_2 \\ 10x_2 - 10x_3 = 10 & \varrho_3 \end{array}$$

$$\xrightarrow{\varrho_3 - 5\varrho_2 \mapsto \varrho_3} \boxed{\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 30x_3 = -30 \end{array}}$$

Solve:
$x_3 = -1$,
$2x_2 + 8 = 8$ or $2x_2 = 0$. $x_2 = 0$
$x_1 - 0 - 1 = 0$. $x_1 = 1$

TWO crucial properties
① same soln as starter system ② easy to solve.

8. Example 2:
$$\begin{array}{rrcrcrcl} & & & x_2 & - & 4x_3 & = & 8 \\ -2(& 2x_1 & - & 3x_2 & + & 2x_3 & = & 1) \\ & 4x_1 & - & 8x_2 & + & 12x_3 & = & 1 \\ & -4x_1 & & +6x_2 & & -4x_3 & = & -2 \end{array} \xrightarrow[\varrho_2 \leftrightarrow \varrho_1]{\varrho_3 - 2\varrho_1 \mapsto \varrho_2} \begin{array}{r} 2x_1 - 3x_2 + 2x_3 = 1 \\ (x_2 - 4x_3 = 8) \cdot 2 \\ -2x_2 + 8x_3 = -1 \\ 2x_2 - 8x_3 = 16 \end{array}$$

$$\xrightarrow{\varrho_3 + 2\varrho_2 \mapsto \varrho_3} \begin{array}{r} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0 = 15 \end{array}$$ ← No solution

No solution.

ANS: No solution.

9. Example 3:
$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 2x_1 & - & 2x_2 & - & 2x_3 & = & 9 \end{array} \xrightarrow[\varrho_2 \leftrightarrow \varrho_1]{\varrho_3 - \varrho_2 \mapsto \varrho_3} \begin{array}{r} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ x_2 - 4x_3 = 8 \end{array}$$

$$\xrightarrow{\varrho_3 - \varrho_2 \mapsto \varrho_3} \boxed{\begin{array}{r} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0 = 0 \end{array}}$$

→ Pick $x_3 = 0$, then $x_2 = 8$, $x_1 = \frac{25}{2}$

Pick $x_3 = -1$, then $x_2 = 4$, $x_1 = \frac{15}{2}$

etc ...

Given x_3,
$x_2 = 8 + 4x_3$
$x_1 = 1 + 3x_2 - 2x_3$

Many solutions