Section One.I.2: Describing the Solutions Set (aka aesthetics)

Goals: (1) Reframe SoLE (and their solutions) in terms of matrices (vectors), (2) Review elementary vector notation and operations.

How we solved a SoLE in Section One.I.1

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ 5x_1 \phantom{{}- 2x_2} - 5x_3 = 10 \end{cases} \xrightarrow{\rho_3 - 5\rho_1 \mapsto \rho_3} \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ \phantom{x_1 - {}} 10x_2 - 10x_3 = 10 \end{cases} \xrightarrow{\rho_3 - 5\rho_2 \mapsto \rho_3} \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ \phantom{x_1 - 2x_2 + {}} 30x_3 = -30 \end{cases}$$

Conclude: $x_3 = -1,\ x_2 = 0,\ x_1 = 1$ via back substitution.

How we solved a SoLE in Section One.I.2

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ 5x_1 \phantom{{}- 2x_2} - 5x_3 = 10 \end{cases} \qquad\qquad \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{x_1 - {}} 2x_2 - 8x_3 = 8 \\ \phantom{x_1 - 2x_2 + {}} 30x_3 = -30 \end{cases} \quad \text{Solve as before.}$$

$$\Downarrow \qquad\qquad\qquad\qquad\qquad\qquad \Uparrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & 8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \xrightarrow{\rho_3 - 5\rho_1 \mapsto \rho_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & 8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \xrightarrow{\rho_3 - 5\rho_2 \mapsto \rho_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & 8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Example 1: Solve the SoLE $\begin{cases} w \phantom{{}+2x} + y + 2z = 0 \\ w + 2x + y + 6z = 8 \\ -w + 2x + 2y + 2z = 20 \end{cases}$ by converting to matrices.

$$\left[\begin{array}{cccc:c} 1 & 0 & 1 & 2 & 0 \\ 1 & 2 & 1 & 6 & 8 \\ -1 & 2 & 2 & 2 & 20 \end{array}\right] \begin{array}{l} e_2 - e_1 \mapsto e_2 \\ e_3 + e_1 \mapsto e_3 \end{array} \left[\begin{array}{cccc:c} 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 4 & 8 \\ 0 & 2 & 3 & 4 & 20 \end{array}\right] e_3 - e_2 \mapsto e_3 \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 4 & 8 \\ 0 & 0 & 3 & 0 & 12 \end{bmatrix}$$

$$w + y + 2z = 0$$
$$2x + 4z = 8$$
$$3y = 12$$

$\boxed{y = 4}$

$\boxed{x = 4 - 2z}$

$\boxed{z = \text{anything}}$

$w = -y - 2z$

$\boxed{w = -4 - 2z}$

Solution:

$(w, x, y, z) = (-4 - 2z, 4 - 2z, 4, z),\ z \in \mathbb{R}$

Check..?

$z = 0,$

$z = 1$

Component wise

Vector Review

Vector

Scalar

- vector versus scalar? $\vec{v}=(1,2,3)$ or $\vec{u}=\begin{pmatrix}0\\-1\\1\\2\end{pmatrix}$, $c=\sqrt{5}$
- vector addition: $\vec{v}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{v}_2=\begin{pmatrix}0\\-1\\4\end{pmatrix}$, $\vec{v}_1+\vec{v}_2=\begin{pmatrix}1+0\\2-1\\3+4\end{pmatrix}=\begin{pmatrix}1\\1\\7\end{pmatrix}$

  (requires the same dimensions!) $\vec{v}_1+\vec{u}$ doesn't make sense!
- scalar multiplication:

$$\sqrt{5}\cdot\vec{v}_1=\sqrt{5}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}1\cdot\sqrt{5}\\2\cdot\sqrt{5}\\3\cdot\sqrt{5}\end{pmatrix}=\begin{pmatrix}\sqrt{5}\\2\sqrt{5}\\3\sqrt{5}\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\sqrt{5}$$

Return to Example 1. Write its solution in vector form

$$\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\begin{pmatrix}-4-2z\\4-2z\\4\\z\end{pmatrix}=\begin{pmatrix}-4\\4\\4\\0\end{pmatrix}+\begin{pmatrix}-2z\\-2z\\0\\z\end{pmatrix}=\begin{pmatrix}-4\\4\\4\\0\end{pmatrix}+\begin{pmatrix}-2\\-2\\0\\1\end{pmatrix}z$$

Final Answer:

$$\left\{\begin{pmatrix}-4\\4\\4\\0\end{pmatrix}+\begin{pmatrix}-2\\-2\\0\\1\end{pmatrix}z \;:\; \underline{z\in\mathbb{R}}\right\}$$

"{ }" mean "set"

means z can take the value of any real number.

z is a "free" variable.

free

$$\text{echelon form } A=\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b\\ \hline 1 & 2 & 0 & 1 & 1 & 5\\ 0 & 2 & -4 & 0 & 2 & -6\\ 0 & 0 & 0 & 0 & 1 & 4\end{array}$$

. Find the solution set of the SoLE.

back substitute

-8-6

$\boxed{x_5=4}$

$2x_2-4x_3+2x_5=-6$

$x_2-2x_3+x_5=-3$

$x_2=2x_3-x_5-3$

$\boxed{x_2=2x_3-7}$

$\boxed{x_3=\text{anything}}$

$\boxed{x_4=\text{anything}}$

$x_1+2x_2+x_4+x_5=5$

$x_1=-2x_2-x_4-x_5+5$

$x_1=-2(2x_3-7)-x_4-4+5$

$\boxed{x_1=-4x_3-x_4+15}$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}-4x_3-x_4+15\\2x_3-7\\x_3\\x_4\\4\end{pmatrix}=\begin{pmatrix}15\\-7\\0\\0\\4\end{pmatrix}+\begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix}x_4+\begin{pmatrix}-4\\2\\1\\0\\0\end{pmatrix}x_3$$

Solution Set:

$$\left\{\begin{pmatrix}15\\-7\\0\\0\\4\end{pmatrix}+\begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix}x_4+\begin{pmatrix}-4\\2\\1\\0\\0\end{pmatrix}x_3 \;:\; x_4,x_3\in\mathbb{R}\right\}$$