Fact: There are many different echelon forms of a SoLE (matrix).

We observed this as a class on Friday's quiz!!!

Today's Ideas: Gauss-Jordan Reduction, which consists of adding some steps after achieving echelon form, results in a unique matrix. Moreover, the reversible nature of elementary row operations can be used to establish that the nature of solution sets of SoLE are not dependent upon its reduced form.

Example of Gauss-Jordan Reduction

Given a SoLe: $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 2x_2 - 8x_3 = 8 \\ 5x_1 \quad\quad - 5x_3 = 10 \end{cases}$ in matrix form $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$

Step 1: Put the matrix into echelon form. Recall this is a left-to-right, top-to-bottom process.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \xrightarrow[\rho_3 - 5\rho_1 \mapsto \rho_3]{} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \xrightarrow[\rho_3 - 5\rho_2 \mapsto \rho_3]{} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Step 2: Make all leading coefficients 1. (Done in one step.)

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix} \xrightarrow[\frac{1}{30}\rho_3 \mapsto \rho_3]{\frac{1}{2}\rho_2 \mapsto \rho_2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Step 3: Use leading 1's to eliminate *all* nonzero entries in that column. This is a right-to-left, bottom-to-top process.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[\rho_2 - 4\rho_3 \mapsto \rho_2]{\rho_1 - \rho_3 \mapsto \rho_1} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[\rho_1 + 2\rho_2 \mapsto \rho_1]{} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

pivots and *pivoting*:

↳ leading terms in row echelon form

↳ using leading terms to eleminate nonzero terms in same column

definition: A matrix or SoLe is in **reduced row echelon form** if

- it's in echelon form (stair steps down)
- all leading terms are 1's
- leading 1's are the only nonzero term in that column.

Example: Use Gauss-Jordan Reduction to put the SoLE $\begin{cases} w - 3x \quad\quad + \ z = 5 \\ -w + \ \ x \quad\quad + 5z = 2 \\ \quad\quad\ \ x + y + \ \ z = 0 \end{cases}$ into reduced row echelon form and solve.

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 5 \\ -1 & 1 & 0 & 5 & 2 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r_2 + r_1 \mapsto r_2} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 5 \\ 0 & -2 & 0 & 6 & 7 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r_3 + \frac{1}{2} r_2 \mapsto r_3} \left[\begin{array}{ccccc} 1 & -3 & 0 & 1 & 5 \\ 0 & -2 & 0 & 6 & 7 \\ 0 & 0 & 1 & 4 & 3.5 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2} r_2 \to r_2} \left[\begin{array}{ccccc} 1 & -3 & 0 & 1 & 5 \\ 0 & 1 & 0 & -3 & -3.5 \\ 0 & 0 & 1 & 4 & 3.5 \end{array}\right] \xrightarrow{r_1 + 3r_2 \to r_1} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -8 & -5.5 \\ 0 & 1 & 0 & -2 & -3.5 \\ 0 & 0 & 1 & 4 & 3.5 \end{array}\right] \end{array}$$

$$\left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8z - 5.5 \\ 2z - 3.5 \\ -4z + 3.5 \\ z \end{pmatrix} = \begin{pmatrix} -5.5 \\ -3.5 \\ 3.5 \\ 0 \end{pmatrix} + \begin{pmatrix} 8 \\ 2 \\ -4 \\ 1 \end{pmatrix} z \ , \ z \in \mathbb{R} \right\}$$

Notes:

Lemma 1.5

- Row operations are reversible.

  $\boxed{*c} \longleftrightarrow \boxed{*\frac{1}{c}}$ b/c $c \neq 0$.

  $\boxed{r_i \leftrightarrow r_j} \longleftrightarrow \boxed{r_i \leftrightarrow r_j}$

  $\boxed{r_i + c r_j \to r_i} \longleftrightarrow \boxed{r_i - c r_j \mapsto r_i}$

- Consequence: If $\boxed{\text{matrix } M_1} \xrightarrow[\text{row operations}]{\text{elementary}} \boxed{\text{matrix } M_2}$,

  then there must be row operations so that

  $\boxed{\text{matrix } M_2} \xrightarrow{\text{row ops}} \boxed{\text{matrix } M_1}$

1.7 Defn

Two matrices (M_1, M_2) are row equivalent if there exist elementary row operations transforming M_1 to M_2 and this is an equivalence relation.] Lemma 1.6

- symmetric
- reflexive
- transitive

Rough sense of "equals".

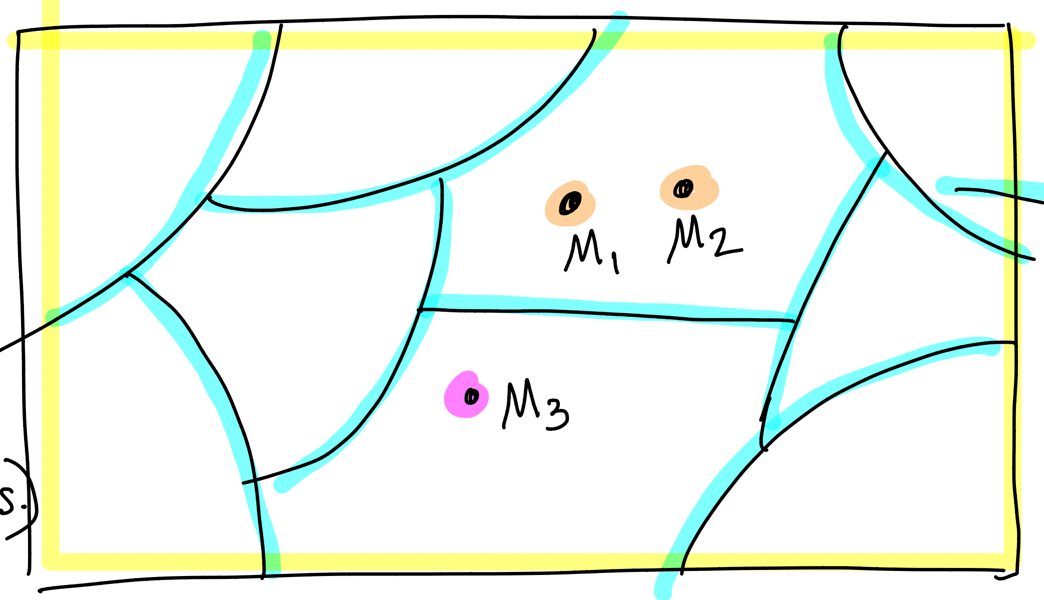
- Two matrices are "the same" if they are row equivalent.
- Two SoLE with property that their matrix forms are row equivalent must have same solution set.

b/c elementary row operations preserve solutions!!

Another view of equivalence relation

M_1 + M_2 are row equivalent (ie same soln).

M_1 + M_3 (M_2 + M_3) are not row equivalent (i.e. different solutions)



The universe of all matrices

partitions (separators) that split matrices into row-equivalent subsets.