We (or the book) proved that the following are vector spaces:

- $V_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,:\, y = 2x \right\}$ under regular vector addition and scalar multiplication.
- $V_2 = \{f : \mathbb{R} \to \mathbb{R} \,:\, f(x) + 3f'(x) = 0\}$ under regular function addition and scalar multiplication.
- $V_3 = \{a_2x^2 + a_1x + a_0 \,:\, a_2, a_1, a_0 \in \mathbb{R}\}$ under regular polynomial addition and scalar multiplication.

Explain why the following examples are *not* vector spaces. Try to find as many reasons as you can.

1. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,:\, y = x + 2 \right\}$ under regular vector addition and scalar multiplication.

- not closed under scalar multiplication: $10 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$ but $30 \neq 10+2$
- not closed under vector addition: $\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ but $4 \neq 0+2$
- No additive identity. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in $V$.

2. $V = \{f : \mathbb{R} \to \mathbb{R} \,:\, f(x) + 3f'(x) = 1\}$ under regular function addition and scalar multiplication.

- not closed under vector addition. $f+g+3(f+g)'$
  $= f + 3f' + g + 3g' = 1+1 = 2 \neq 1$
- not closed under scalar multiplication: $10 \cdot f + 3(10f)' = 10(f+3f')$
  $= 10\,(1) = 10$
  $\neq 1$
- no additive identity: $z + 3z' = 0+0 = 0 \neq 1$
  $z(x) = 0$

3. $V = \{a_2x^2 + a_1x + a_0 \,:\, a_2, a_1, a_0 \in \mathbb{Z}\}$ under regular polynomial addition and scalar multiplication.
   The symbol $\mathbb{Z}$ denotes all integers: $\ldots -2, -1, 0, 1, 2, 3, 4, \ldots$

- not closed under scalar multiplication:
  $\pi\,(7x^2 + 3x + 4) = 7\pi x^2 + 3\pi x + 4\pi$ but
  $7\pi \notin \mathbb{Z}$.

**Lemma 1.16** In any vector space $V$ and for any $\vec{v} \in V$ and $r \in \mathbb{R}$,

- $0\vec{v} = \vec{0}$
- $-1 \cdot \vec{v} + \vec{v} = \vec{0}$
- $r \cdot \vec{0} = \vec{0}$

proof:
$$\begin{aligned} \vec{v} &= 1 \cdot \vec{v} && (10) \\ &= (1+0)\vec{v} && (\text{prop. of } \mathbb{R}) \\ &= 1\cdot\vec{v} + 0\vec{v} && (7) \\ &= \vec{v} + 0\vec{v} && (10). \end{aligned}$$

Now, $\vec{v} + \vec{w} = \vec{v} + \vec{w} + 0\vec{v}$, (2),(3),(5) $\vec{w}$ inverse

So $\vec{0} = 0\vec{v}$, ✓

SECTION TWO.I.2: SUBSPACES AND SPANNING SETS

**Definition:** Let V be a vector space. A subset W of V is a <u>subspace of V</u> if W is itself a vector space.

**Example:** Let $V = \mathbb{R}^3$, the vector space of 3-dimensional real-valued vectors under the usual vector and scalar operations. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$. Show $W$ is a subspace of $V$.

- W inherits properties 2, 3, 7–10 from V. We still need to check closure (1, 6) and the presence of identity + inverses (4, 5).
- If $x+y-z=0$ and $x'+y'-z'=0$, then $x+x'+y+y'-(z+z')=0$.

  So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix} \in V$.
- If $x+y-z=0$, then $r(x+y-z) = rx+ry-rz=0$.

  So $r\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix} \in V$.
- $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in V$ since $0+0-0=0$.
- If $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V$, then $\begin{bmatrix} -x \\ -y \\ -z \end{bmatrix} \in V$ since $-x-y+z = -(x+y-z) = -0 = 0$.

Note to me: Go back and look at $V_1, V_2, V_3$ at top of page 1.

**Example:** Let $V = \mathbb{R}^3$, the vector space of 3-dimensional real-valued vectors under the usual vector and scalar operations. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$. Show $W$ is a subspace of $V$.

There is a <u>different</u> way to demonstrate W is a subspace of V: as a solution set of a system of linear equations (!)

SOLE or matrix

$x + y - z = 0$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}$$
$$\begin{array}{cccc} & \uparrow & \uparrow & \\ & \text{free} & \text{free} & \end{array}$$

already in rref!

Solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y+z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z$

Answer: $\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z \;:\; y, z \in \mathbb{R} \right\} = W}$

or in words: W is the set of all linear combinations of the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Thus, it must be a subspace of $\mathbb{R}^3$.