Example: Do Gauss-Jordan reduction on the matrix below but record the steps as linear combinations of rows.

$$\begin{bmatrix}1&2&1\\-1&2&0\\3&0&8\end{bmatrix}\xrightarrow{r_1+r_2\mapsto r_2}\begin{bmatrix}1&2&1\\0&4&1\\3&0&8\end{bmatrix}\xrightarrow{r_3-3r_1\mapsto r_3}\begin{bmatrix}1&2&1\\0&4&1\\0&-6&5\end{bmatrix}\xrightarrow{r_3+\frac{3}{2}r_2\mapsto r_3}\begin{bmatrix}1&2&1\\0&4&1\\0&0&\frac{13}{2}\end{bmatrix}$$

$$\begin{bmatrix}\vec{r_1}\\\vec{r_2}\\\vec{r_3}\end{bmatrix}\xrightarrow{r_1+r_2\mapsto r_2}\begin{bmatrix}\vec{r_1}\\\vec{r_1}+\vec{r_2}\\\vec{r_3}\end{bmatrix}\xrightarrow{r_3-3r_1\mapsto r_3}\begin{bmatrix}\vec{r_1}\\\vec{r_1}+\vec{r_2}\\\vec{r_3}-3\vec{r_1}\end{bmatrix}\xrightarrow{r_3+\frac{3}{2}r_2\mapsto r_3}\begin{bmatrix}\vec{r_1}\\\vec{r_1}+\vec{r_2}\\(\vec{r_3}-3\vec{r_1})+\frac{3}{2}(\vec{r_1}+\vec{r_2})\end{bmatrix}$$

Observation: Every linear combination of a 3-dimensional row vector gives a 3-dimensional row vector. Nothing bad happens.

definition A *vector space* of $\mathbb{R}$ consists of a set V along with two operations: $+$ and $\cdot$ such that for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all $r, s \in \mathbb{R}$ all of the following ten conditions hold:

1. V is closed under vector addition: For every $\vec{u}, \vec{v} \in V$, $\vec{u}+\vec{v} \in V$
2. Vector addition is commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
3. Vector addition is associative: $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
4. V has an additive identity: There is some $\square \in V$ so that $\vec{u}+\square=\vec{u}$
5. V has additive inverses: For every $\vec{u} \in V$ there is some $\vec{v} \in V$, so that $\vec{u}+\vec{v}=\square$
6. V is closed under scalar multiplication: For every $r \in \mathbb{R}, \vec{v} \in V$, $r\cdot\vec{v} \in V$.
7. Scalar multiplication distributes over scalar addition: For all $r, s \in \mathbb{R}$, $\vec{v} \in V$, $(r+s)\vec{v} = r\vec{v}+s\vec{v}$
8. Scalar multiplication distributes over vector addition: For all $r \in \mathbb{R}$, $\vec{v}, \vec{u} \in V$ $r(\vec{v}+\vec{u}) = r\vec{v}+r\vec{u}$
9. Scalar multiplication is associative: $(rs)\vec{v} = r(s\vec{v})$
10. The scalar number 1 acts as a multiplicative identity: $1\cdot\vec{v}=\vec{v}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}, \quad r\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} rx_1 \\ ry_1 \end{bmatrix}$$

Demonstrate that the following are vector spaces.

Example 1: $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 2x \right\}$ under regular vector addition and scalar multiplication.

(1) $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ 2x_1+2x_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ 2(x_1+x_2) \end{bmatrix}$ ← still in V.

(2) $\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} x_2+x_1 \\ y_2+y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

(3) follows from vector properties

(4) $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{v} + \vec{0} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{v}$

(5) Given $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, pick $\vec{u} = \begin{bmatrix} -x \\ -y \end{bmatrix}$. So $\vec{v} + \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -x \\ -2x \end{bmatrix} = \begin{bmatrix} -x \\ 2(-x) \end{bmatrix}$

nothing to do with $y=2x$

(6) $\vec{v} = \begin{bmatrix} x \\ 2x \end{bmatrix}$, $r\vec{v} = \begin{bmatrix} rx \\ r2x \end{bmatrix} = \begin{bmatrix} rx \\ 2(rx) \end{bmatrix}$ ← still in V

⋮

(10) $1 \cdot \vec{v} = 1 \cdot \begin{bmatrix} x \\ 2x \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = \vec{v}$

Example 2: $V = \{f : \mathbb{R} \to \mathbb{R} \ : \ f(x) + 3f'(x) = 0\}$ under regular function addition and scalar multiplication.

(1) $f, g \in V$; So $f + 3f' = 0$ and $g + 3g' = 0$.

Now $(f+g) + 3(f+g)' = f + g + 3(f' + g') = f + 3f' + g + 3g' = 0 + 0 = 0$

So $f+g$ is in V.

(2) $f+g = g+f$ (3) $(f+g)+h = f+(g+h)$

(4) $f + \square = f$? $\square$ is zero fcn $g(x) = 0$

(5) $f + \square = \vec{0}$. So $\square = -f$. And $-f + 3(-f)' = -(f + 3f') = 0$

(6) $rf + 3(rf)' = r(f + 3f') = r \cdot 0 = 0$

(7) (8) (9) follow from knowledge of calculus

(10)