Example: Do Gauss-Jordan reduction on the matrix below but record the steps as linear combinations of rows.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \\ 3 & 0 & 8 \end{bmatrix} \xrightarrow[r_1 + r_2 \mapsto r_2]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 8 \end{bmatrix} \xrightarrow[r_3 - 3r_1 \mapsto r_3]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & -6 & 5 \end{bmatrix} \xrightarrow[r_3 + \frac{3}{2}r_2 \mapsto r_3]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & \frac{13}{2} \end{bmatrix}$$

$$\begin{bmatrix} \vec{r_1} \\ \vec{r_2} \\ \vec{r_3} \end{bmatrix} \xrightarrow[r_1 + r_2 \mapsto r_2]{} \begin{bmatrix} \vec{r_1} \\ \vec{r_1} + \vec{r_2} \\ \vec{r_3} \end{bmatrix} \xrightarrow[r_3 - 3r_1 \mapsto r_3]{} \begin{bmatrix} \vec{r_1} \\ \vec{r_1} + \vec{r_2} \\ \vec{r_3} - 3\vec{r_1} \end{bmatrix} \xrightarrow[r_3 + \frac{3}{2}r_2 \mapsto r_3]{} \begin{bmatrix} \vec{r_1} \\ \vec{r_1} + \vec{r_2} \\ (\vec{r_3} - 3\vec{r_1}) + \frac{3}{2}(\vec{r_1} + \vec{r_2}) \end{bmatrix}$$

Observation: Every linear combination of a 3-dimensional row vector gives a 3-dimensional row vector. Nothing bad happens.

definition A *vector space* of $\mathbb{R}$ consists of a set V along with two operations: $+$ and $\cdot$ such that for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all $r, s \in \mathbb{R}$ all of the following ten conditions hold:

1. V is closed under vector addition:
2. Vector addition is commutative:
3. Vector addition is associative:
4. V has an additive identity:
5. V has additive inverses:
6. V is closed under scalar multiplication:
7. Scalar multiplication distributes over scalar addition:
8. Scalar multiplication distributes over vector addition:
9. Scalar multiplication is associative:
10. The scalar number acts as a multiplicative identity:

Demonstrate that the following are vector spaces.

Example 1: $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;:\; y = 2x \right\}$ under regular vector addition and scalar multiplication.

Example 2: $V = \{f : \mathbb{R} \to \mathbb{R} \;:\; f(x) + 3f'(x) = 0\}$ under regular function addition and scalar multiplication.