Example Again: Let $V = \mathbb{R}^3$, the vector space of 3-dimensional real-valued vectors under the usual vector and scalar operations. Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$. Show W is a subspace of V.

SoLE

$x + y - z = 0$

or matrix

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}$$

already in rref!

free free

Solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y+z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z$

Answer: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} z : y, z \in \mathbb{R} \right\} = W$

or in words: W is the set of all linear combinations of the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Thus, it must be a subspace of $\mathbb{R}^3$.

Lemma 2.9: V is a vector space. S is a nonempty subset of the set V. The following are equivalent.

- S is a vector space
- For every $r_1, r_2 \in \mathbb{R}$, $\vec{v}_1, \vec{v}_2 \in S$, $r_1\vec{v}_1 + r_2\vec{v}_2 \in S$
- S is closed under all possible linear combinations of vectors from S

definition: Let S be a nonempty subset of the vector space V. The *span of S* (or $[S]$ or *linear closure of S*) is

$$[S] = \{ c_1\vec{s}_1 + c_2\vec{s}_2 + \dots + c_n\vec{s}_n : c_1, c_2, \dots, c_n \in \mathbb{R}, \vec{s}_1, \vec{s}_2, \dots, \vec{s}_n \in S \}$$

= the set of all possible linear combinations of vectors of S

Sample Problems

1. Let $S = \{(1,1,0),(0,1,0)\}$ be a subset of $\mathbb{R}^3$, the vector space of all real-valued 3-vectors under the usual vector addition and scalar multiplication. Which of the vectors below are in $span(S)$? (Show your work.)

$\vec{v} = (8,-10,0)$ $= (8,8,0)+(0,-18,0) = 8(1,1,0) - 18(0,1,0)$

$\vec{w} = (1,2,3)$. Not in span(S). The vectors in S have zeros in the 3rd coordinate.

$\vec{x} = (a,b,0)$

$= (a,a,0)+(0,b-a,0) = a(1,1,0)+(a-b)(0,1,0)$

Observe that S is a subspace of $\mathbb{R}^3$; it's the xy-plane.

2. How would you know if the set

$$T = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right\}$$

spanned the set of all 2×2 matrices?

Need to find a_1, a_2, a_3, a_4, so that

$$a_1\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + a_3\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + a_4\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

So
$$\begin{aligned} a_1 &= b_1 \\ 2a_2 &= b_2 \\ a_1 + a_2 + a_3 + a_4 &= b_3 \\ a_3 + 2a_4 &= b_4 \end{aligned}.$$

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, for any $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, there is a solution. So S spans the set of all 2x2 matrices.

Lemma 2.15:

In a vector space, span(S) is always a vector space.