1. Below are several *subsets* of $V = \mathbb{R}^3$. Which ones span $\mathbb{R}^3$? Are some more efficient than others?

   (a) $A = \{(1,1,0),(0,1,0)\}$

   doesn't span

   on previous sheet, we established that $[A] \neq \mathbb{R}^3$

   (b) $B = \{(1,0,0),(0,1,0),(0,0,1)\}$, $[B] = \mathbb{R}^3$ because

   spans

   for every $(a,b,c) \in \mathbb{R}^3$, $a(1,0,0) + b(0,1,0) + c(0,0,1) = (a,b,c)$.

   (c) $C = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0)\}$ $[C] = \mathbb{R}^3$ because ① $B \subseteq C$ and $[B] = \mathbb{R}^3$.

   spans

   less efficient

   or ② for every $(a,b,c) \in \mathbb{R}^3$, $a(1,0,0) + b(0,1,0) + c(0,0,1) = (a,b,c)$.

   or ③ for every $(a,b,c) \in \mathbb{R}^3$, $a(1,1,0) + (b-a)(0,1,0) + c(0,0,1) = (a,b,c)$.

   (d) $D = \{(0,1,0),(0,0,1),(1,1,0)\}$ $[D] = \mathbb{R}^3$ because

   spans

   for every $(a,b,c) \in \mathbb{R}^3$, $a(1,1,0) + (b-a)(0,1,0) + c(0,0,1) = (a,b,c)$.

2. (Intuitive) **Definition:** Let S be a subset of the vectors in the vector space V. We say S is **linearly independent** if no vector in S can be written as a linear combination of other vectors in S.
   Otherwise, we say S is linearly dependent

   trivial → What does this mean?
   - Don't count $\vec{s}_i = c\,\vec{s}_i$.
   - Don't count all zero constants: $0\vec{s}_1 + 0\vec{s}_2 + \dots + 0\vec{s}_k = 0\vec{s}_n$

3. Determine if the set $T = \{\vec{u} = (1,2,0), \vec{v} = (1,1,1), \vec{w} = (1,3,-1)\}$ of vectors in $\mathbb{R}^3$ are linearly independent.

   Observation: $2(1,2,0) + (-1)(1,1,1) = (2-1, 4-1, -1) = (1,3,-1)$
   So $2\vec{u} - \vec{v} = \vec{w}$. So T is linearly DEpendent.

   Note, $\vec{w}$ can be written in terms of $\vec{u}$ and $\vec{v}$. But also $\vec{u}$ can be written in terms of $\vec{v}$ and $\vec{w}$: $\vec{u} = \frac{1}{2}\vec{v} + \frac{1}{2}\vec{w}$.
   or even: $\vec{v} = 2\vec{u} - \vec{w}$.

Think first!

4. Determine if the set $S = \{\vec{s_1} = (1,2,1,1), \vec{s_2} = (1,1,1,1), \vec{s_3} = (3,4,0,-1), \vec{s_4} = (0,8,-1,4)\}$ of vectors in $\mathbb{R}^4$ are linearly independent.

Solve $c_1\vec{s}_1 + c_2\vec{s}_2 + c_3\vec{s}_3 + c_4\vec{s}_4 = \vec{0}$

$c_1(1,2,1,1) + c_2(1,1,1,1) + c_3(3,4,0,-1) + c_4(0,8,-1,4) = (0,0,0,0)$

$$\begin{aligned} c_1 + c_2 + 3c_3 \quad &= 0 \\ 2c_1 + c_2 + 4c_3 + 8c_4 &= 0 \\ c_1 + c_2 \quad - c_4 &= 0 \\ c_1 + c_2 - c_3 + 4c_4 &= 0 \end{aligned}$$

or

$$\left[\begin{array}{cccc:c} 1 & 1 & 3 & 0 & 0 \\ 2 & 1 & 4 & 8 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

linearly independent

Lemma 1.5 $S = \{\vec{s_1}, \vec{s_2}, \vec{s_3}, \cdots, \vec{s_n}\}$ is a subset of the vector space V.

real definition.

S is linearly independent if and only if the only solution to $c_1\vec{s}_1 + c_2\vec{s}_2 + c_3\vec{s}_3 + \dots + c_n\vec{s}_n = \vec{0}$ is

$c_1 = c_2 = c_3 = \dots = c_n = 0$

$\boxed{\text{Proof}}$ Restate.

Q] Can $\vec{0} \in S$ if S is linearly independent?

5. V is a vector space and $S \subseteq V$, $\vec{v} \in V$. What can you conclude if $[S \cup \{\vec{v}\}] = [S]$? Can your reverse this implication?

So adding $\vec{v}$ doesn't change the span? Then $\vec{v} \in [S]$.

Lemma: $[S \cup \{\vec{v}\}] = [S]$ if and only if $\vec{v} \in [S]$.

$\vec{v} = c_1\vec{s}_1 + c_2\vec{s}_2 + \dots + c_n\vec{s}_n$

6. V is a vector space and $S \subseteq V$, $\vec{s} \in S$. What can you conclude if $[S - \{\vec{s}\}] = [S]$? Can you reverse this implication?

You can remove $\vec{v}$ from S and not change span S? $\vec{v}$ must be a linear combination of elements of S.

Lemma: $[S - \{\vec{v}\}] = [S]$ if and only if $\vec{v}$ can be written as a linear combination of $S - \vec{v}$

7. Let S be a subset of the vector space V. If, for every $\vec{v} \in S$, $[S - \vec{v}] \neq [S]$ (that is, the subspace $[S - \vec{v}]$ is smaller than the space $[S]$) what can you conclude about S? Does the reverse implication still hold?

Every time an element of S is removed, the span of the set gets smaller? Then S is linearly independent.

Lemma: $[S - \vec{v}] \neq [S]$ for every $\vec{v} \in S$ if and only if S is linearly independent.

equivalent to 6.

Extra Notes

Lemma Let S be a linearly independent set.
Let $\vec{v} \notin \vec{S}$.
$S \cup \{\vec{v}\}$ is also linearly independent
if and only if

$$\boxed{\vec{v} \notin [S]}$$

Pf: $S \cup \{\vec{v}\}$ linearly dependent if and only if $\vec{v} \in [S]$

Equivalent.

Lemma: Let $S \subseteq V$, S a finite subset of vectors from V.
Then there exists a finite, linearly independent subset of S, say T, so that $[T] = [S]$.

Let A be a matrix w/ rreform $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$

Are the rows of B linearly independent? (No)
Are the nonzero rows of B linearly independent? (Yes)
What can you say about the rows of matrix A?
They must have been linearly DEpendent!