1. **Example:** Find the eigenvalues associated with the matrix $A = \begin{pmatrix} 4 & 1 \\ -8 & -5 \end{pmatrix}$

Find λ so that there is a $\vec{v}$ so that $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I_2)\vec{v} = \vec{0}$ has a nontrivial soln. So, we need λ so that $A - \lambda I_2$ is singular.

or $\det(A - \lambda I_2) = 0$.

So $\begin{vmatrix} 4-\lambda & 1 \\ -8 & -5-\lambda \end{vmatrix} = (4-\lambda)(-5-\lambda) + 8 = -20 + \lambda + \lambda^2 + 8 = \lambda^2 + \lambda - 12 = (\lambda+4)(\lambda-3) = 0$

So $\underline{\lambda = 3, -4}$

2. **Definition/Lemma 3.9 §5.2.3:** A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the *characteristic equation* $det(A - \lambda I_n) = 0$. The expression $det(A - \lambda I_n)$ is called the *characteristic polynomial* of A.

3. **Example:** Find the characteristic polynomial and all eigenvalues associated with the matrix $B = \begin{pmatrix} -2 & 3 & 4 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

$B - \lambda I_3 = \begin{pmatrix} -2-\lambda & 3 & 4 \\ 0 & 5-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix}$

$\det(B - \lambda I_3) = (-2-\lambda)(5-\lambda)(+1-\lambda)$ ← characteristic poly

eigenvalues: $\lambda = -2, 5, +1$

4. **Example:** Construct a 4×4 matrix with eigenvalues $0, -1, 7, 7$.

$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & 4 & 7 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix} = C$

5. **Example:** Are the previous examples singular or nonsingular matrices?

nonsingular: A, B

singular: C

6. **Recall:** Interchanging rows of a matrix changes its determinant by $-1 \cdot$(original det)

Replacing a row with itself plus a multiple of an another row changes the determinant not at <u>all</u>.

7. **Observation:** It is possible to take any square matrix A, use *only* the two row operations from item #6, and find an upper-triangular row equivalent matrix B such that

$$det(A) = (-1)^r \det(B) = (-1)^r \cdot (\text{product of the main diagonal of } B),$$

where r is the number of row exchanges.

8. **Example:** Assume $B = \begin{pmatrix} -2 & 3 & 4 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is obtained form A by the following row operations: (i) exchange rows 1 and 3; (ii) add 2·(row 1) to row 3; (ii) add $-2\cdot$ row 1 to row 2. Find $det(A)$.

$$A \xleftrightarrow{r_1 \leftrightarrow r_3} A' \xrightarrow{r_3+2r_1 \to r_3} A'' \xrightarrow{-2r_1+r_2 \to r_2} B$$

$$\det(A) = (-1)\det B = (-1)(-2)(5)(1) = 10$$

9. **Theorem:** The square matrix A is nonsingular if and only if $\lambda = 0$ is not an eigenvalue.

10. **Definition 1.2 §5.2.1:** Let A and B be $n \times n$ matrices. We say A **is similar to** B if there exists an invertible matrix P such that $P^{-1}AP = B$.

11. **Example:** Show that $A = \begin{pmatrix} -13 & 21 \\ 7 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 0 \\ 0 & -20 \end{pmatrix}$ are similar. (Hint: Use $P = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$.

Check: $P^{-1}AP = B$. OR Check $AP = PB = \begin{bmatrix} 8 & -60 \\ 8 & 20 \end{bmatrix}$

Notes:
- $A = PBP^{-1}$
- So $A^k = (PBP^{-1})^k = PB^kP^{-1} = P\begin{pmatrix} 8^k & 0 \\ 0 & (-20)^k \end{pmatrix}P^{-1}$
- Where did P even come from? Find...

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -13+21 \\ 7+1 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix} = 8\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -39-21 \\ 21-1 \end{pmatrix} = \begin{pmatrix} -60 \\ 20 \end{pmatrix} = -30\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$