We could say "A and B are similar to each other!"

1. **Definition 1.2 §5.2.1:** Let A and B be $n \times n$ matrices. We say A **is similar to** B if there exists an invertible matrix P such that $P^{-1}AP = B$.

2. **Example:** Show that $A = \begin{pmatrix} -13 & 21 \\ 7 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 0 \\ 0 & -20 \end{pmatrix}$ are similar. (Hint: Use $P = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$.)

$P^{-1} = \begin{pmatrix} 1/4 & 3/4 \\ 1/4 & -1/4 \end{pmatrix}$; Check $P^{-1}AP = \begin{pmatrix} 8 & 0 \\ 0 & -20 \end{pmatrix}$

3. Think of some alternate ways to write the equation from Definition 1.2.

$AP = PB$ → In practice, this can sometimes be slightly faster to check

$A = PBP^{-1}$

$A = Q^{-1}BQ$ → where $Q = P^{-1}$ → "is similar to" is a symmetric relationship.

4. **Example:** Show that $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not similar to $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Show that no nonsingular matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the property that

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\equiv$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$ $\equiv$ $a=b$ $b=-a$ $c=d$ $d=-c$

So $a=b=c=d=0$

5. What would be easier to find by hand, A^{10} or B^{10}?

$A^{10} = \begin{pmatrix} -13 & 21 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} -13 & 21 \\ 7 & 1 \end{pmatrix}\cdots\begin{pmatrix} -13 & 21 \\ 7 & 1 \end{pmatrix}$

$B^{10} = \begin{pmatrix} 8^{10} & 0 \\ 0 & (-20)^{10} \end{pmatrix}$. But wait!! $A^{10} = (P^{-1}BP)^{10} = P^{-1}B^{10}P$ (!!!)

6. Let $\vec{v_1}$ and $\vec{v_2}$ be the columns of P. Find $A\vec{v_1}$ and $A\vec{v_2}$.

$\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\vec{v_2} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$A\vec{v_1} = \begin{pmatrix} -13 & 21 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -13+21 \\ 7+1 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix} = 8\vec{v_1}$

$A\vec{v_2} = \begin{pmatrix} -13 & 21 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -39-21 \\ 21-1 \end{pmatrix} = \begin{pmatrix} -60 \\ 20 \end{pmatrix} = -20\vec{v_2}$

7. Give two distinct arguments to explain how you know $\{\vec{v_1}, \vec{v_2}\}$ form a basis for $\mathbb{R}^2$.

- way 1: P is invertible ⇒ cols are lin indep.
- way 2: $\vec{v_1}$ + $\vec{v_2}$ have distinct eigenvalues.
- way 3: $\vec{v_1}$ + $\vec{v_2}$ are lin independent by inspection!

8. **Definition:** A square matrix is diagonalizable if A is similar to a diagonal matrix.

So A is diagonalizable.

$\to P^{-1}AP = D \leftarrow$ diagonal!

9. **Theorem:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Moreover

Define P to be the $n \times n$ matrix such that its columns consist of n linearly independent eigenvectors of A. Then

$$A = PDP^{-1} \quad \text{OR} \quad P^{-1}AP = D \leftarrow \text{a diagonal matrix.}$$

where D is a diagonal matrix such that the entries on the main diagonal are the eigenvalues associated with the eigenvectors of the columns of P.

10. **Example from Worksheet on Monday 7 Nov §3.5.1 & 3.5.2:**

Define $h : \mathbb{R}^3 \to \mathbb{R}^3$ by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$. ← Clearly a linear map.

How did we get this?

(a) Its matrix representation of h with respect to the standard basis is: $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = A$

How?

(b) For basis $B = \left\langle \vec{b_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{b_2} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \vec{b_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$, the change of basis matrix from basis B to the standard basis is $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix} = \operatorname{Rep}_{B,\mathcal{E}_3}(id) = P^{-1}$

How?

(c) The change of basis matrix from the standard basis to B is $\begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/6 & 1/6 & -1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \operatorname{Rep}_{\mathcal{E}_3,B}(id) = P$

How?

(d) The matrix representation of h with respect to basis B is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$ ← diagonal

11. Observations?

- $$\begin{tikzcd} \mathbb{R}^3_{\mathcal{E}_3} \arrow[r, "A"] & \mathbb{R}^3_{\mathcal{E}_3} \arrow[d, "P"] \\ B \arrow[u, "P^{-1}"] \arrow[r, "D"] & B \end{tikzcd}$$

$$D = P^{-1}AP$$

We were demonstrating A & D are similar; That A is diagonalizable.

- Where did the vectors of B come from?
They are eigenvectors w/ assoc. eigenvalues $\lambda = -1, \lambda = -1, \lambda = 2$.

12. Return to Example from #10.

Define $h : \mathbb{R}^3 \to \mathbb{R}^3$ by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$.

- Find the matrix of the linear transformation, say A.
- Find the characteristic polynomial of A and use it to find any eigenvalues of A.
- For each eigenvalue, find a basis for the corresponding eigenspace.
- Show that A is diagonalizable.

- $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (images of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in each respective column.)

- $\det(A - \lambda I_3) = \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix}$

$= -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda) = -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda$

$= -\lambda^3 + 3\lambda + 2 = -(\lambda + 1)^2 (\lambda - 2)$

- $(A + 1\lambda) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Soln to homog.: $x + y + z = 0$ or $x = -y - z$.

$\begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; basis $\left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$.

$(A - 2\lambda) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$; Soln: $x = z$, $y = z$

$\begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

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- Use eigenvectors to find P:

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix};$$

$$\text{Find } P^{-1} = \begin{pmatrix} -1/3 & 2/3 & 1/3 \\ 1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Show A is diagonalizable:

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \leftarrow \text{a diagonal matrix w/ eigenvalues!} \smile$$