DIAGONALIZATION

1. **Definition 1.2 §5.2.1:** Let A and B be $n \times n$ matrices. We say A **is similar to** B if there exists an invertible matrix P such that $P^{-1}AP = B$.

2. **Example:** Show that $A = \begin{pmatrix} -13 & 21 \\ 7 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 0 \\ 0 & -20 \end{pmatrix}$ are similar. (Hint: Use $P = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$.)

3. Think of some alternate ways to write the equation from Definition 1.2.

4. **Example:** Show that $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not similar to $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

5. What would be easier to find by hand, A^{10} or B^{10}?

6. Let $\vec{v_1}$ and $\vec{v_2}$ be the columns of P. Find $A\vec{v_1}$ and $A\vec{v_2}$.

7. Give two distinct arguments to explain how you know $\{\vec{v_1}, \vec{v_2}\}$ form a basis for $\mathbb{R}^2$.

8. **Definition:** A square matrix is diagonalizable if A is similar to a diagonal matrix.

9. **Theorem:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

   Define P to be the $n \times n$ matrix such that its columns consist of n linearly independent eigenvectors of A. Then

   $$A = PDP^{-1}$$

   where D is a diagonal matrix such that the entries on the main diagonal are the eigenvalues associated with the eigenvectors of the columns of P.

10. **Example from Worksheet on Monday 7 Nov §3.5.1 & 3.5.2**:

    Define $h : \mathbb{R}^3 \to \mathbb{R}^3$ by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$.

    (a) Its matrix representation of h with respect to the standard basis is: $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

    (b) For basis $B = \left\langle \vec{b_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{b_2} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \vec{b_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$, the change of basis matrix from basis B to the standard basis is $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$

    (c) The change of basis matrix from the standard basis to B is $\begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/6 & 1/6 & -1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$.

    (d) The matrix representation of h with respect to basis B is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

11. Observations?

12. Return to Example from #10.

Define $h : \mathbb{R}^3 \to \mathbb{R}^3$ by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$.

- Find the matrix of the linear transformation, say A.
- Find the characteristic polynomial of A and use it to find any eigenvalues of A.
- For each eigenvalue, find a basis for the corresponding eigenspace.
- Show that A is diagonalizable.