1. **Definition:** The vector $\vec{v}$ is an eigenvector of matrix A with associated eigenvalue λ means

$$A\vec{v} = \lambda \vec{v}$$

2. **Observation:** If $\vec{v}$ is an eigenvector of matrix A with associated eigenvalue λ and k is an positive integer, then $A^k\vec{v} =$ $\lambda^k \vec{v}$

So they form a basis.

3. **Observation:** If A is an 3×3 matrix with 3 linearly independent eigenvectors, $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ associated with eigenvalues λ_1, λ_2, λ_3, and k is a positive integer, find an easy way to write $A^k\vec{x}$ for any $\vec{x} \in \mathbb{R}^3$.

So $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

So $A\vec{x} = A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3)$

$= c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3$

$= c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + c_3\lambda_3\vec{v}_3$.

So $A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + c_3\lambda_3^k\vec{v}_3$

4. Let O_k and R_k denote the owl and rat populations at month k where O_k counts the number of owls and R_k is measured in thousands of rats. Suppose a model describing these populations is below:

$$O_{k+1} = (0.5)O_k + (0.4)R_k \text{ and } R_{k+1} = (-0.104)O_k + (1.1)R_k.$$

   (a) Assume a population begins with 10 owls and $10,000$ rats in month 0, determine how many owls and rats the model indicates in month 1.

$$\begin{array}{cc} O & R \\ \end{array}$$
$$\begin{pmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{pmatrix}\begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 9 \\ 9.96 \end{pmatrix}$$

$\vec{x} = \begin{pmatrix} O_k \\ R_k \end{pmatrix}$ population vector

9 owls and 9,960 rats a month later

(b) In month 2? In month 3?

$$\begin{pmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{pmatrix}\begin{pmatrix} 9 \\ 9.96 \end{pmatrix} = \begin{pmatrix} 8.4840 \\ 10.0200 \end{pmatrix}; \quad \begin{pmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{pmatrix}\begin{pmatrix} 8.4840 \\ 10.02 \end{pmatrix} = \begin{pmatrix} 8.25 \\ 10.1397 \end{pmatrix}$$

(c) What are the eigenvalues and associated eigenvectors associated with the matrix in part (a)?

$\vec{v}$	$\begin{pmatrix} 10 \\ 13 \end{pmatrix} = \vec{v}_1$	$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \vec{v}_2$
λ	1.02	0.58

eig(A)

(d) Write an arbitrary population vector from month k, $\vec{x_k}$, with respect to the eigenvectors from part c and use this to determine $\vec{x_{k+1}}$, the population vector in month $k+1$.

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 \; ; \quad A = \begin{pmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{pmatrix}$$

$$\vec{x}_{k+1} = A^k\vec{x} = c_1(1.02)^k\,\vec{v}_1 + c_2(0.58)^k\,\vec{v}_2$$

(e) What happens as $k \to \infty$?

as $k \to \infty$, $(0.58)^k \to 0$

So, long-term, $\vec{x}_{k+1} \approx c_1(1.02)^{k+1}\,\vec{v}_1$

$$= (1.02)\cdot c_1(1.02)^k\,\vec{v}_1$$

$$\approx (1.02)\,\vec{x}_k$$

So, longterm, both owls + rats grow at a 2% rate per month.
Longterm the ratio of owls to rats steadies at 10 : 13,000.

5. A different owl population is modeled by the discrete dynamical system

$$\begin{pmatrix} j_{k+1} \\ s_{k+1} \\ a_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{pmatrix} \begin{pmatrix} j_k \\ s_k \\ a_k \end{pmatrix}$$

where k is measured in years and $\vec{x_k} = \begin{pmatrix} j_k \\ s_k \\ a_k \end{pmatrix}$ represents the number of female juvenile, subadult and adult owls.

(a) Assume the matrix $\begin{pmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{pmatrix}$ has three distinct eigenvalues each with magnitude less than 1. What can you conclude about the long term trajectory of the owl population?

The population is approaching zero.

Since each term

$\vec{X}_k = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2 + c_3 \lambda_3^k \vec{v}_3$ will approach zero.

(b) On the other hand, the matrix $\begin{pmatrix} 0 & 0 & 0.33 \\ 0.3 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{pmatrix}$ has three distinct eigenvalues one of which is 1.01 with eigenvector $\begin{pmatrix} 10 \\ 3 \\ 31 \end{pmatrix}$. The other two eigenvalues still have magnitude less than 1. What can you conclude about the long term trajectory of the owl population?

The owl population will grow at a 1%/year rate longterm.