1. **Definition 3.5 §5.2.3:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. A scalar λ is called an **eigenvalue** of A if there exists a nontrivial solution to the equation $A\vec{x} = \lambda\vec{x}$. In this case, we say $\vec{x}$ is an eigenvector associated with eigenvalue λ.

2. Example 1: Let $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$.

* quick check: $\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+6 \\ 5+2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   (a) Show that $\vec{v} = \begin{pmatrix} 3 \\ -5/2 \end{pmatrix}$ is an eigenvector but $\vec{w} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ is not.

$A\vec{v} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -5/2 \end{pmatrix} = \begin{pmatrix} 3 - 15 \\ 15 - 5 \end{pmatrix} = \begin{pmatrix} -12 \\ 10 \end{pmatrix} = -4\begin{pmatrix} 3 \\ -5/2 \end{pmatrix} = -4\vec{v}$

$A\vec{w} = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 - 30 \\ 15 - 10 \end{pmatrix} = \begin{pmatrix} -27 \\ 5 \end{pmatrix} \overset{?}{=} k\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ Then $-27 = k \cdot 3$, $5 = k(-5)$

So $k = -1$ and $k = -9$.

   (b) Show that 7 is an eigenvalue of A.

Find $\vec{v}$ so that $A\vec{v} = 7\vec{v}$ or $A\vec{v} = 7 \cdot I_2\vec{v}$ or $A\vec{v} - 7I_2\vec{v} = \vec{0}$ or $(A - 7I_2)\vec{v} = \vec{0}$. Now, $A - 7I_2 = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}$.

Solve $\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So $-x + y = 0$. OR $x = y$.

So $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector w/ associated eigenvalue 7. *

3. **Definition 3.1.2 §5.2.3:** For $n \times n$ matrix A. The set of all eigenvectors associated with eigenvalue λ forms a subspace of $\mathbb{R}^n$ and is called the **eigenspace** associated with eigenvalue λ.

How do we know? Check closure under + & scalar mult.

4. Example 2: Let $A = \begin{pmatrix} -1 & -2 & 5 \\ 2 & -5 & 5 \\ 2 & -2 & 2 \end{pmatrix}$, a matrix with eigenvalue -3. Find a basis for the corresponding eigenspace.

Find a basis for all $\vec{v}$ so that $A\vec{v} = -3\vec{v}$. OR Find solution set for $(A + 3I_3)\vec{x} = \vec{0}$. Now $A + 3I_3 = \begin{pmatrix} -1 & -2 & 5 \\ 2 & -5 & 5 \\ 2 & -2 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 5 \\ 2 & -2 & 5 \\ 2 & -2 & 5 \end{pmatrix}$.

So $\left(\begin{array}{ccc:c} 2 & -2 & 5 & 0 \\ 2 & -2 & 5 & 0 \\ 2 & 5 & 5 & 0 \end{array}\right) \xrightarrow{\text{ref}} \left(\begin{array}{ccc:c} 1 & -1 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. So $x - y + \frac{5}{2}z = 0$ or $x = y - \frac{5}{2}z$.

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From $x = y - \frac{5}{2}z$, we get:

$$\text{Soln. set} = \left\{ \begin{pmatrix} y - \frac{5}{2}z \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\}$$

$$= \left\{ y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -5/2 \\ 0 \\ 1 \end{pmatrix} : y, z \in \mathbb{R} \right\}$$

$$= \text{span}\underbrace{\left(\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/2 \\ 0 \\ 1 \end{pmatrix} \right\} \right)}_{*}$$

A quick check: $A = \begin{pmatrix} -1 & -2 & 5 \\ 2 & -5 & 5 \\ 2 & -2 & 2 \end{pmatrix}$

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 5 \\ 2 & -5 & 5 \\ 2 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1-2 \\ 2-5 \\ 2-2 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \checkmark$$

$$A \cdot \begin{pmatrix} -5/2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 5 \\ 2 & -5 & 5 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} -5/2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} + 5 \\ -5 + 5 \\ -5 + 2 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ 0 \\ -3 \end{pmatrix} = -3\begin{pmatrix} -5/2 \\ 0 \\ 1 \end{pmatrix} \checkmark$$

5. **Theorem:** If A is triangular, then its eigenvalues are the entries on its main diagonal.

6. Example: Construct a 3×3 matrix A with eigenvalues $0, -1$, and 5 and find an eigenvector associated with each eigenvalue.

Pick $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{pmatrix}$.

vector		assoc. e-value
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\xrightarrow{\text{easy}}$	0
$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$	$\xrightarrow{\text{actual work}}$	-1
$\begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$	$\xrightarrow{\text{actual work}}$	5

Strategy

Solve $\left(\begin{array}{ccc|c} -5 & 1 & 1 & 0 \\ 0 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{ccc|c} -5 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

7. **Theorem 3.18 §5.2.3:** Let A be an $n \times n$ matrix with eigenvectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}$ associated with distinct eigenvalues $\lambda_1, \lambda_2 \cdots, \lambda_k$. Then the set of eigenvectors $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is linearly independent. Why? $v_1 = c_2\vec{v}_2 + c_3\vec{v}_3 + \dots + c_k\vec{v}_k$, what is $A\vec{v}_1$?

8. Question: Does the previous theorem really need the eigenvalues to be distinct? Yes. $\vec{v}_1$ and $2\vec{v}_1$. both have λ_1 as e-value.

9. Question: If $\vec{v}$ is an eigenvector of matrix A associated with eigenvalue λ, can you draw any conclusions about eigenvectors and/or eigenvalues for matrix A^2?

$A\vec{v} = \lambda\vec{v}$.

$A^2\vec{v} = A(A\vec{v}) = A\cdot(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$.

So $\vec{v}$ is an eigenvector of A assoc. w/ eigenvalue λ^2.

10. Question: How would you go about finding *all* eigenvalues associates with matrix A.?

Find all λ, so that $(A - \lambda I_n)\vec{v} = \vec{0}$ has a non-trivial soln. So we need $A - \lambda I_n$ to be singular.
So we need $\det(A - \lambda I_n) = 0$. (!!)