Observe that the document has spaced out the problems so that there space for comments.

Chapter 3 Problems

1. Prove the right and left cancellation laws for a group G, stated formally below.

   Let G be a group and suppose $a, b, c \in G$.

   - Prove that if $ab = ac$, then $b = c$.
   - Prove that if $ba = ca$, then $b = a$.

   Proof:

2. Let G be a group and suppose that for every $a, b \in G$, $(ab)^2 = a^2b^2$. Prove that G must be abelian.

   Proof:

3. List all the subgroups of the symmetry group of an equilateral triangle.

   List:

Chapter 4 Problems

4. All questions below are about $U(10)$, the group of units of $\mathbb{Z}_{10}$. (Recall that the definition of $U(n)$ and an example $U(8)$ are on page 43 of our text.)

   (a) List the elements of $U(10)$.
   List:

   (b) Show that $U(10)$ is cyclic.
   Answer:

   (c) Find all generators of $U(10)$.
   Answer:

   (d) Find an element $a \in U(10)$, such that $a \neq 1$ and $a^2 = 1$.
   Answer:

   (e) Prove that there always exists some $a \in U(n)$ such that $a \neq 1$ and $a^2 = 1$.
   Proof:

5. Prove or disprove the following statements.

   (a) All the generators of $\mathbb{Z}_{60}$ are prime.
   Answer:

   (b) $U(8)$ is cyclic.
   Answer:

   (c) $(\mathbb{Q},+)$ is cyclic.
   Answer:

   (d) If every proper subgroup of a group G is cyclic, then G itself must be cyclic.
   Answer:

   (e) If the group G has a finite number of subgroups, then G is finite.
   Answer:

6. Observe that $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ are elements of $GL_2(\mathbb{R})$. Find $|A|$, $|B|$ and $|AB|$.

   (You are strongly encouraged to use technology here!)

   What curious thing do you observe?

   Answer:

7. Prove that if p is prime, then $\mathbb{Z}_p$ has no nontrivial, proper subgroups.

   Proof:

8. Let $\sigma = (1\,4\,3\,7\,6)$ and $\tau = (1\,5\,6)(2\,3\,4)$.

   (a) Express $\sigma\tau$ and $\tau\sigma$ as a cycle or a product of disjoint cycles.
   Answer:

   (b) Express σ and τ as a product of transpositions and identify each as even or odd. **Answer:**

   (c) Find σ^{-1}. Write is as a cycle or a disjoint product of cycles. **Answer:**

   (d) Explain why any permutation of $\sigma \in S_n$ can be written as the product of at most $n-1$ transpositions. **Answer:**

9. Show that A_{10} contains an element of order 15. (Note that it is not enough to find such an element. You need to both be certain it is in A_{10} and demonstrate how you know its order.)