For each problem below, G is a group with subgroups H and K.

1. (a) List all possible (disjoint) cycle structures of A_5. (Here are some of the questions you should be asking yourself. Can A_5 have a permutation consisting of single 5-cycle? 4-cycle? 3-cycle? ... Can A_5 have a permutation consisting of a 3-cycle and a (disjoint) 2-cycle?...)
   Answer:

   (b) List all possible (disjoint) cycle structures of A_6.
   Answer:

2. All questions below are about D_5, the 5th dihedral group.

   (a) Write out all the elements of D_5 using permutation notation. (Assume the letters $\{1,2,3,4,5\}$ are in cyclic order along the regular 5-gon.)
   Answer:

   (b) Make a specific choice of r and s such that every element of D_5 can be written in the form $s^a r^b$ in that order for appropriate choice of a and b.
   Answer:

   (c) Using your choice of r and s above write every element of D_5 in the form $r^b s^a$ in that order for appropriate choice of a and b.
   Answer:

3. Let G be a group and let $a \in G$. Define $f_a(x) : G \to G$ by $f(x) = ax$. We claim f is a permutation of G.

   (a) Assume $G = U(9) = \{1, 2, 4, 5, 7, 8\}$ and $a = 8$. Describe $f_8(x)$, the permutation of $U(9)$ determined by 8, using cycle notation.
   Answer:

   (b) Prove that for any group G and any $a \in G$, $f_a(x)$ is a permutation of G. (You should **start** by remembering the definition of a permutation.)
   Proof:

4. For each group G and subgroup H, identify all the left and right cosets of H in G. Use the notation we used in class. It is sufficient to simple state them. You do not need to give an explanation of your work.

   (a) $G = \mathbb{Z}$, $H = 3\mathbb{Z}$
   Answer:

   (b) $G = \mathbb{Z}_{12}$, $H = \langle 4 \rangle$
   Answer:

   (c) $G = S_4$, $H = A_4$
   Answer:

   (d) $G = S_4$, $H = D_4$
   Answer:

   (e) $G = S_4$, $H = \{(), (123), (132)\}$ (Find left cosets only.)
   Answer:

   (f) Give an example of a group G and subgroup H of G such that H will have an infinite number of left cosets in G.
   Answer:

5. Let G be a group, H a subgroup of G, and $g_1, g_2 \in G$. Prove each implication below **using first principles**. This means you can use only definitions. You cannot use Lemma 6.3. (You are proving part of Lemma 6.3 in this problem.)

   (a) Prove that if $g_1H \subseteq g_2H$, then $g_1H = g_2H$.
   Proof:

   (b) Prove that $g_1H = g_2H$ if and only if $g_1^{-1}g_2 \in H$. (You may use part (a) from this problem.)
   Proof:

6. Let G be a group and H a subgroup of G. Prove that if $ghg^{-1} \in H$, for every $g \in G$ and for every $h \in H$, then $gH = Hg$ for all $g \in G$. (This is, under the condition $ghg^{-1} \in G$, left and right cosets are the same. Give a careful argument here.)
   Proof:

7. Suppose that $[G:H]=2$. Prove that for every $a,b \in G \backslash H$, $ab \in H$.
   Proof:

8. Prove that if $[G:H]=2$, then $gH = Hg$ for every $g \in G$.
   Proof: