1. **Definition:** The **center** of a group is

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}.$$

That is, an element of G is in the center of G if it commutes will all other elements in the group. Find $Z(G)$ for each group below. Bald answers are ok here.

(a) $G = (\mathbb{Z}, +)$
Answer:

(b) $G = S_3$
Answer:

(c) $G = D_6$
Answer:

2. Lemma 6.3 tells us that for $H \leq G$, a group and for any $g_1, g_2 \in G$ the statement $g_1H = g_2H$ is equivalent to $Hg_1^{-1} = Hg_2^{-1}$. The Lemma does **not** say $g_1H = g_2H$ is equivalent to $Hg_1 = Hg_2$. Why? Either this second equivalence is **implied** or it is false. If it is implied, then prove it. If it is false, prove that.

Conclusion:

Proof of Conclusion:

3. Give a detailed proof that every cyclic group of order n is isomorphic to $(\mathbb{Z}_n, +)$.

   Proof:

4. Prove or disprove: $U(8) \cong \mathbb{Z}_4$

   Proof:

5. Let $G = \mathbb{R}\backslash\{-1\}$ with binary operation $*$ defined as

$$a * b = a + b + ab.$$

Recall that in Homework 2, you proved that G is a group with this operation. Prove that $(G, *) \cong (\mathbb{R}^*, \cdot)$.

Proof:

6. Find the order of each of the following elements. Bald answers are acceptable here.

(a) $(3,4)$ in $\mathbb{Z}_4 \times \mathbb{Z}_6$
Answer:

(b) $(6,15,4)$ in $\mathbb{Z}_{30} \times \mathbb{Z}_{43} \times \mathbb{Z}_{24}$
Answer:

(c) $(5,10,15)$ in $\mathbb{Z}_{25} \times \mathbb{Z}_{25} \times \mathbb{Z}_{25}$
Answer:

(d) $(8,8,8)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{24} \times \mathbb{Z}_{80}$
Answer:

7. (a) Find nontrivial subgroups H and K of $U(9)$ such that $U(9)$ is an internal direct product of H and K. Prove your answer is correct.
   Answer: $H = \{\}, K = \{\}$
   Proof:

   (b) Theorem 9.27 says that if G is an internal direct product of H and K, then G is **isomorphic** to the external direct product $H \times K$. Explicitly write all the elements in the group $H \times K$ along with the associcated group operation(s). Then state explicitly an isomorphism between G and $H \times K$.
   Group Elements: $H \times K = \{\}$
   Isomorphism: Your f goes here.

8. Prove that D_4 cannot be the internal direct product of two of its proper subgroups.
   Proof:

9. Let G be a group and $g \in G$. Define the function $f_g(x) : G \to G$ by $f_g(x) = gxg^{-1}$. Prove that f_g is an isomorphism from G to itself. (Such a function is called an **automorphism** of G.)
   Proof:

10. Prove that if $G \cong G'$ and $H \cong H'$, then $G \times H \cong G' \times H'$.

    Proof: