1. (a) Write out the elements of $G = \mathbb{Z}_6$ and $H = \mathbb{Z}_2 \times \mathbb{Z}_3$ and then find a function, ϕ, that demonstrates that $G \cong H$ and show that your function is an isomorphism.

   Elements of $\mathbb{Z}_6$:
   Elements of $\mathbb{Z}_2 \times \mathbb{Z}_3$:
   Proof that $G \cong H$:

   (b) Prove that $G = \mathbb{Z}_3 \times \mathbb{Z}_4$ and $H = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic.
   Proof:

2. For each of the following groups G, determine whether H is a normal subgroup of G. If H is a normal subgroup, write out a Cayley table for the factor group of G/H.

   (a) $G = S_4, H = A_4$.

   Answer:

   Here is a sample Cayley table for $U(8)$. In the LaTeXcode, the symbol & separates the column entries in a row.

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

   (b) $G = A_5, H = \{(), (123), 132)\}$. (Hint: Use a Theorem from this section.)

   Answer:

   (c) $G = S_4, H = D_4$.

   Answer:

   (d) $G = \mathbb{Z}, H = 5\mathbb{Z}$.

   Answer:

3. (a) Find all subgroups of D_4.
   Answer:

   (b) Identify which of the subgroups of D_4 are normal.
   Answer:

   (c) Describe each of the **nonisomorphic** factor groups of D_4.
   Answer:

4. Let T be the group of 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. The group operation is **matrix multiplication**. Let U consist of all matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x \in \mathbb{R}$.

   (a) By example, confirm that T is not abelian.

   Answer:

   (b) Given $C = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T$, find C^{-1} and confirm that it is in T.

   Answer:

   (c) Prove $U \leq T$.
   Proof:

   (d) Prove U is abelian.
   Proof:

   (e) Prove $U \triangleleft T$.
   Proof:

   (f) Prove T/U is abelian.
   Proof:

   (g) Is T normal in $GL_2(\mathbb{R})$? Justify your answer.
   Answer:

5. Show that the intersection of two normal subgroups is normal.
   Proof:

6. Define the **centralizer** of an element g in a group G to be the set $C(g) = \{x \in G : xg = gx\}$.

   (a) Find $C(\rho)$, where $\rho = (12)$ and $G = S_4$.
   Answer:

   (b) Prove that $C(g)$ is a subgroup of G.
   Proof: